Examination of Various Models of Viscoelasticity

Tim Foldy-Porto
Yale University
Neuromuscular Biomechanics
timothy.foldy-porto@yale.edu

***Abstract*—We explore the Maxwell model for viscoelasticity with the aim of evaluating its efficacy in describing the properties of animal muscle. While it succeeds in modeling the two-regime frequency response of real muscle, we find that the Maxwell model fails to accurately describe muscle's step response, which can be characterized by stretched exponential decays. In light of this, we propose methods for modifying the Maxwell model such that stretched exponentials solve the governing differential equations. Additionally, we find that a fractional differential operator is necessary for the production of stretched exponential solutions; we discuss methods for dealing with such operators, with applications to control theory and robotics.**

I. Introduction

Animal muscle can, in one sense, be characterized by its response to perturbations. Key parts of animals' ability to achieve stable legged locomotion can be attributed to the passive response of their muscles, which can be much faster than any conscious response [1]. It is of great interest to control engineers to understand this perturbation response so as to replicate the properties of animal muscles for applications in robotics [2]. However, even state-of-the-art robotic actuators fail to achieve the force, stiffness, and power generation capabilities of real muscle.

There have been many mechanical models proposed for muscle, a material which can be classified as *viscoelastic* due its exhibition of both viscous and elastic characteristics when undergoing deformation. The two simplest models of viscoelasticity that still encapsulate some of the intricacies of muscle's response are the Voigt model (a spring and a damper in parallel) and the Maxwell model (a spring and a damper in series) [3]. While the Voigt model has achieved success in modeling passive tissues [4], it fails to describe muscle's ability to dissipate elastic stresses over time [5], and we need not spend time examining it.

The Maxwell model, on the other hand, is a serious candidate for describing the properties of muscle: it succeeds in capturing the nature of real muscle in both the low-frequency and high-frequency perturbation regimes. However, a recent study has shown that, upon a step perturbation in length, the force response of muscles can be characterized by *stretched exponentials*, which are of the form $F = \exp(-t^{\beta})$. It is believed that this feature of muscle contributes to animals' ability to stabilize themselves over long time periods. The Maxwell model notably cannot produce such exponentials.

In this paper, we examine the properties of the Maxwell model—first in the time domain and then in the Laplace domain—with the aim of seeing why exactly it fails to produce a stretched exponential response. We then explore methods of modifying the Maxwell model in the hope of producing a more muscle-like response; this takes us into the realm of fractional calculus.

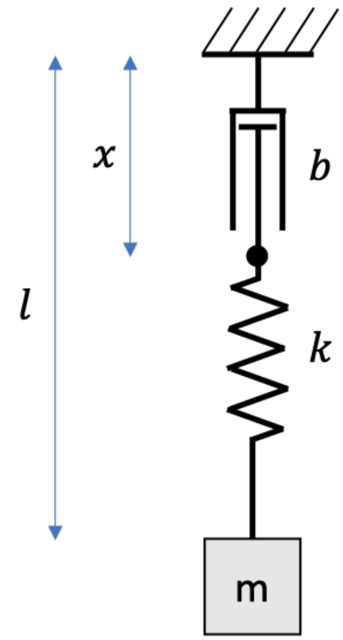

Fig. 1. Scenario of consideration: a mass m suspended from a Maxwell material (spring and damper in series).

II. Maxwell Materials

The Maxwell model for a viscoelastic material is a spring and a damper in series, shown in figure 1. The key finding of this section is that Maxwell materials **do not** yield stretched-exponential behavior and therefore do not capture one of the key features of muscle dynamics. Nevertheless, it is interesting to examine the governing equations of Maxwell materials in order to see *why* they fail to produce stretched-exponential solutions to a step input. Consider the mechanical system depicted in figure 1: a mass suspended by Maxwell material.

The force balance equations for this system are:

$$\begin{aligned} m\ddot{l} &= mg - k(l-x) \\ b\dot{x} &= k(l-x) \end{aligned} \tag{1}$$

To turn this into a system of coupled first-order differential equations, we make the following definitions definitions:

$$\dot{l} = z$$
$$\dot{z} = g - \gamma(l - x) \qquad (2)$$
$$\dot{x} = \alpha(l - x)$$

where

$$\alpha = \frac{k}{b} \quad \text{and} \quad \gamma = \frac{k}{m} \qquad (3)$$

The system of equations to solve then becomes:

$$\begin{bmatrix} \dot{l} \\ \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\gamma & 0 & \gamma \\ \alpha & 0 & \alpha \end{bmatrix} \begin{bmatrix} l \\ z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \qquad (4)$$

Mathematica can solve this inhomogeneous system of equations fairly easily; the resulting solutions are essentially long sums of exponentials that depend on g, γ, and α that are not worth writing explicitly here. One key observation, however, is that nowhere in the solutions is there a stretched exponential, meaning that no matter how one tweaks the parameters it is impossible to generate stretched exponential behavior with a Maxwell material.

In order to get a better sense of the dependence of the system dynamics on the parameters, it is useful to plug in values for various combinations of g, γ, and α.

A. The baseline solution

To get a sense of the behavior of this system, we first set all parameters to one:

$$g = \alpha = \gamma = 1 \qquad (5)$$

In this case, the position l as a function of time t looks like:

$$l_0(t) = \left(t - \frac{2}{3}e^{-t/2}\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \qquad (6)$$

A plot of the baseline solution is shown in figure 2. We see that for short time scales the exponential term dominates, leading to initial oscillations in the length. These oscillations quickly subside (like $e^{-t/2}$ for our selection of parameters), however, and the length soon begins changing like t (at a constant velocity). This can be interpreted physically as the system hitting its terminal velocity. This interpretation is bolstered by the solution to $x(t)$, the position of the top end of the spring. Mathematica gives:

$$x(t) = \left(t - e^{-t/2} \times \text{non-linear terms}\right) \qquad (7)$$

The non-linearities in $x(t)$ subside at the same rate as the non-linearities in $l(t)$, so after an amount of time $t >> 2$ we have $l(t) - x(t) = 0$, meaning the length of the spring is constant and therefore no longer contributes to the evolution of the system. As one would expect, the terminal velocity in the $t >> 2$ regime is governed entirely by b, the damping constant. We should note, however, that 2 is not a special number, but rather an artifact of our specific choice for g, α, and γ.

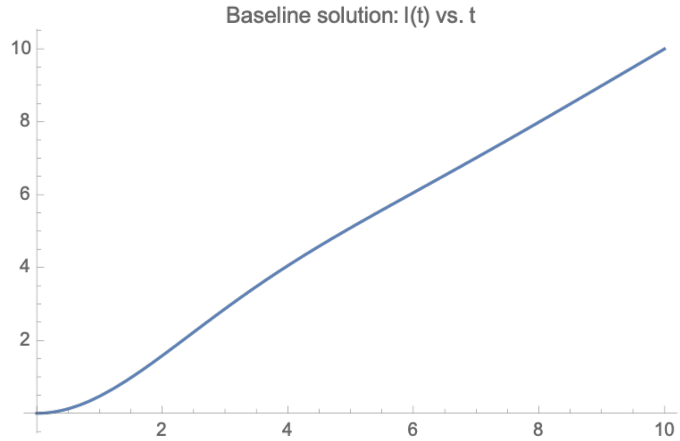

Fig. 2. Plot of $l(t)$, the solution to equation 4 for $g = \alpha = \gamma = 1$. We see that, for short timescales, the length undergoes slight oscillations, and for long timescales the length stretches linearly with time.

B. Dependence of $l(t)$ on g

To determine the dependence of our solution $l(t)$ on g, we have Mathematica solve our system of equations for $\alpha = \gamma = 1$. The resulting solution is:

$$l(t) = g\left(t - \frac{2}{3}e^{-t/2}\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \qquad (8)$$
$$l(t) = g\ l_0(t)$$

This tells us that our system response scales linearly with the strength of the gravitational field, which makes intuitive sense and serves as a nice sanity check. Quadratic or logarithmic dependence on g would make for a very strange world.

C. Dependence of $l(t)$ on γ

When $g = \alpha = 1$,Mathematica returns a very long and messy looking equation. The easiest way to understand how γ affects our system is to plot $l(t)$ for different values of γ and compare visually. This plot is shown in figure 3, which shows a correlation between γ and both the frequency and duration of the transient oscillations in the response. Additionally, it appears as if there is a nonlinear correlation between γ and the magnitude of the response.

While it would be unhelpful to write the exact equation for $l(t)$ here, looking at the general form of the equation elucidates the origin of these correlations. Essentially, the length depends on time and γ like

$$l(t) = \frac{1}{\gamma}\left(t + 1 - \frac{1}{\gamma}\right) + f(\gamma)e^{-t/2}\left(e^{(1+\sqrt{1-4\gamma})} + e^{(-1+\sqrt{1-4\gamma})}\right) \qquad (9)$$

where $f(\gamma)$ has a pole at $\gamma = 0$ and $\gamma = 1/4$. The radical in the exponential tells us that, for $\gamma > 1/4$, the system undergoes oscillations with frequency:

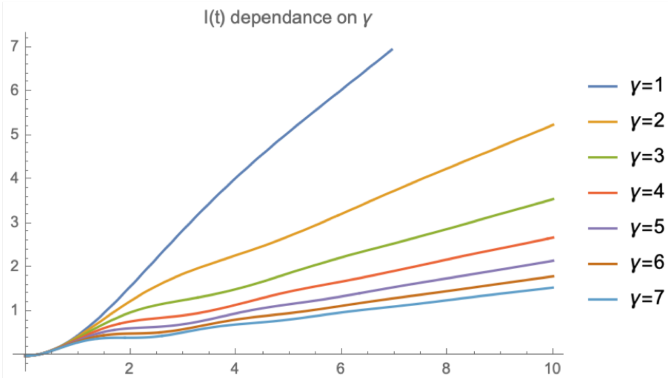

Fig. 3. Plot of $l(t)$, the solution to equation 4, for various values of γ with $g = \alpha = 1$.

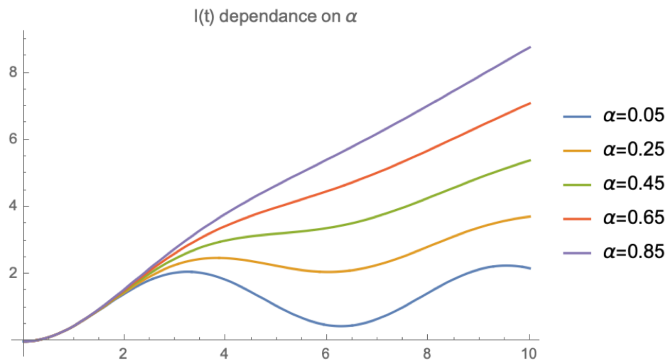

Fig. 4. Plot of $l(t)$, the solution to equation 4, for various values of α with $g = \gamma = 1$.

$$\omega = \frac{1}{2}\sqrt{4\gamma - 1} \tag{10}$$

This explains the frequency of $\sqrt{3}/2$ we saw in the baseline solution $l_0(t)$, where γ was set to one.

One additional point of consideration is to compare our system with other similar systems. A normal spring-mass system has an oscillation frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\gamma} \tag{11}$$

In a parallel spring-mass-damper (SMD) system, the frequency of oscillation is given by:

$$\omega = \frac{1}{2}\sqrt{4\gamma - b^2} \tag{12}$$

We have yet to see our system's dependence on α, which carries the factor of b, but already we can see that in the small t limit our system behaves similarly to a spring and damper in parallel.

D. Dependence of $l(t)$ on α

Having noted the similar γ-dependence between equations 10 and 12, we could naively expect α to affect the *transient* response of our system similar to how it affects the parallel SMD. However, the opposite is the case. When we ask Mathematica to solve our differential equations for $g = 1$ it returns:

$$l(t) = f(\alpha,\gamma)e^{-t/2}\left(e^{(\alpha+\sqrt{\alpha^2-4\gamma})} + e^{(-\alpha+\sqrt{\alpha^2-4\gamma})}\right) \tag{13}$$

Figure 4 shows how $l(t)$ depends on α with $\gamma = 1$. Contrary to the parallel SMD system, smaller values of α (i.e. higher ratios of damping to spring stiffness) yield faster and longer-lasting oscillations in our system (a series SMD).

This observation makes more sense if we imagine the physical picture: for high values of damping compared to stiffness (low α), the mass will only feel the spring, and will therefore undergo normal harmonic oscillations but with slow-timescale perturbations due to the movement of the damper. For high α, the mass will primarily feel the damper, and will fall at terminal velocity but with high-frequency (dependent on γ) perturbations due to the spring.

III. Laplace-Domain Solutions

While it was all fun and good to represent our equations in the time-domain and have Mathematica sort through the ugliness, the dynamics of a mass suspended from a Maxwell material (figure 1) can be solved much more elegantly in the Laplace domain. Our initial equations:

$$\begin{aligned}\ddot{l} &= g - \gamma(l - x)\\ \dot{x} &= \alpha(l - x)\end{aligned} \tag{14}$$

can be transformed into the Laplace domain:

$$\begin{aligned}s^2\hat{l}(s) &= \frac{g}{s} - \gamma(\hat{l}(s) - \hat{x}(s))\\ s\hat{x}(s) &= \alpha(\hat{l}(s) - \hat{x}(s))\end{aligned} \tag{15}$$

These can easily be rearranged to get expressions for $\hat{x}(s)$ and $\hat{l}(s)$ in terms of the fundamental parameters:

$$\begin{aligned}\hat{l}(s) &= \frac{(s+\alpha)g}{s^4 + \alpha s^3 + \gamma s^2}\\ \hat{x}(s) &= \frac{\alpha g}{s^4 + \alpha s^3 + \gamma s^2}\end{aligned} \tag{16}$$

These three equations (14 - 16) produce the same results that Mathematica did but in a much more illuminating form. As a sanity check, we can convert the equation for $\hat{l}(s)$ into the time domain for $g = \alpha = \gamma = 1$. The result is:

$$l(t) = \left(t - \frac{2}{3}e^{-t/2}\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \tag{17}$$

This is identical to equation 6, meaning our Laplace-domain representation is correct.

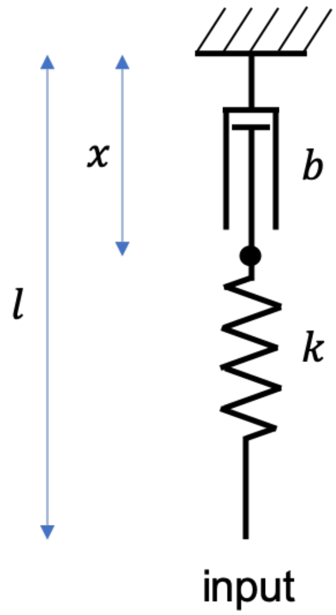

Fig. 5. A Maxwell material subject to an external position source that determines $l(t)$.

A. *Step Response of Maxwell Material*

Next, it is useful to consider the step response of our Maxwell material from figure 1. This result can be compared to the step response of muscle to gain an understanding of the efficacy of our the Maxwell model in describing the properties of muscle. In this case, our scenario is similar to the one studied in section II, but with several simplifying changes. This new system is shown in figure 5.

The mass has been replaced by a position source (infinitely strong and rigid) and the variable of consideration has become $x(t)$, the position of the connection point between the spring and the damper. Typically, one would measure the output force produced by the body to characterize the material. However, we know the force to be $F = k(l(t) - x(t))$, and since we control $l(t)$ for all time, it suffices to solve for the step response of $x(t)$. In contrast to the suspended mass scenario, there is only one relevant equation here:

$$\dot{x} = \alpha(l - x) \tag{18}$$

We'll skip the time-domain nonsense and go straight to the Laplace domain.

$$s\hat{x}(s) = \alpha(\hat{l}(s) - \hat{x}(s)) \tag{19}$$

$$\hat{x}(s) = \frac{\alpha}{s + \alpha}\hat{l}(s) \tag{20}$$

So the transfer function for this system is:

$$\hat{H}(s) = \frac{\alpha}{s + \alpha} \tag{21}$$

To determine the step response, we multiply our transfer function by $\hat{l}(s) = 1/s$:

$$\hat{x}(s) = \hat{H}(s)\frac{1}{s} = \frac{\alpha}{s^2 + \alpha s} \tag{22}$$

Converting to time domain, the solution is:

$$x(t) = 1 - e^{-\alpha t} \tag{23}$$

In terms of the force on the position source, this translates to

$$F = ke^{-\alpha t} \tag{24}$$

for $t > 0$. In equation 24 we have derived the well-known force response of a Maxwell material to a step input. There are two things to note: the force is proportional to the spring constant k, and the force decays to zero with a timescale determined solely by α. The first observation is no surprise, given that springs can be seen as mechanical computers that convert positions to forces, and since $l(t)$ is constant for $t > 0$, the force will be purely proportional to $x(t)$. The second observation makes sense in light of dimensional analysis. We have:

$$[g] = \frac{m}{s^2} \qquad [k] = \frac{kg}{s^2} \qquad [b] = \frac{kg}{s} \tag{25}$$

$$[\alpha] = \frac{[k]}{[b]} = \frac{1}{s} \qquad [\gamma] = \frac{[k]}{[m]} = \frac{1}{s^2} \tag{26}$$

Since α is the only constant with units of frequency, it has to be the constant that appears next to t in exponentials. It is also possible to have a square root of γ appear, as seen in equation 9, but γ is no longer relevant since we switched to the scenario depicted in figure 5.

Having gone through a comprehensive derivation of the properties of a Maxwell material, we can now safely say that the Maxwell model does not encapsulate the stretched exponential response known to characterize real muscle.

IV. Stretched Exponentials

As described in the introduction, it is believed that a stretched-exponential response to a step perturbation in length is one of the defining characteristics of animal muscle. From the perspective of a control theorist who attempts to mimic the properties of muscle for use in robotics applications, the question becomes: what kinds of controllers give rise to stretched-exponential responses? Or, more mathematically, what kinds of differential equations yield stretched exponentials as their solutions?

A. The Naive Approach

One approach that some might call naive (because it is accessible to the mathematical lay-person), is to look for the generating equations of the *Taylor expansion* of a stretched exponential. Unfortunately, we are concerned with solutions that are valid for arbitrarily long amounts of time, in which case a Taylor series is not a simplification. The goal here, however, is to gain insight into what makes a stretched exponential unique, and then use that information to guide our development of its governing differential equations. The solution we hope to generate is:

$$f(t) = 1 - e^{-t^\beta} \quad 0 < \beta < 1 \tag{27}$$

This can be written as:

$$1 - e^{-t^\beta} = 1 - \sum_{k=0} (-1)^k \frac{(t^\beta)^k}{k!} \tag{28}$$

$$= t^\beta - \frac{t^{2\beta}}{2} + \frac{t^{3\beta}}{6} - \dots \tag{29}$$

Additionally, we know the Laplace transform:

$$L\{t^\alpha\} = \frac{\Gamma(1+\alpha)}{s^{1+\alpha}} \tag{30}$$

Where Γ is the gamma function, defined as:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{31}$$

For $z \in \mathbb{Z}$, this reduces to:

$$\Gamma(z) = (z-1)! \tag{32}$$

In our case, we can now represent the stretched-exponential function as:

$$L\{1 - e^{-t^\beta}\} = \frac{1}{s} - \sum_{k=0} (-1)^k \frac{\Gamma(1+k\beta)}{k!\ s^{1+k\beta}} \tag{33}$$

Recall, our goal is to find D, a generalized differential operator, such that:

$$Df(t) = \Theta(t) \tag{34}$$

Where $f(t)$ is a stretched exponential (equation 27) and $\Theta(t)$ is the Heaviside step function. In the Laplace domain, this translates to finding $\hat{H}(s)$ such that:

$$\hat{f}(s) = \hat{H}(s)\frac{1}{s} \tag{35}$$

$\hat{f}(s)$ is plugged in from equation 33, changing the lower limit of the sum so as to get rid of the leading 1, we get:

$$\sum_{k=1} (-1)^k \frac{\Gamma(1+k\beta)}{k!\ s^{1+k\beta}} = \hat{H}(s)\frac{1}{s} \tag{36}$$

$$\hat{H}(s) = \sum_{k=1} (-1)^k \frac{\Gamma(1+k\beta)}{k!\ s^{k\beta}} \tag{37}$$

To a control engineer, this problem is now somewhat solved. All one has to do is devise a system such that the transfer function is the one described by equation 37—easier said than done. In the limiting case that $\beta = 1$, we have:

$$\hat{H}(s) = \sum_{k=1} (-1)^k \frac{\Gamma(1+k)}{k!\ s^k} = \sum_{k=1} (-1)^k \frac{1}{s^k} \tag{38}$$

This is simply a geometric series, which has the well-known solution:

$$\hat{H}(s) = \frac{1}{s+1} \tag{39}$$

Plugged into equation 35:

$$\hat{f}(s) = \frac{1}{s+1}\left(\frac{1}{s}\right) \tag{40}$$

$$s\hat{f}(s) + \hat{f}(s) = \frac{1}{s} \tag{41}$$

Converted back to time domain, we have:

$$\frac{d}{dt}f(t) + f(t) = \Theta(t) \longrightarrow D = \frac{d}{dt} + 1 \tag{42}$$

This corresponds with our analysis of the Maxwell model, which was found to generate exponential decay from the governing equations:

$$\frac{d}{dt}x(t) + \alpha x(t) = \alpha\Theta(t) \tag{43}$$

Based on this limiting cases, we can make the educated guess that for a general β such that $0 < \beta < 1$:

$$D = \frac{1}{\alpha^\beta}\frac{d^\beta}{dt^\beta} + 1 \tag{44}$$

where d^β is some fractional differential operator and $\alpha = k/m$ was reintroduced to make the units work out. Comparing this to equation 18, our new physical system might be described by:

$$\frac{d^\beta}{dt^\beta}x(t) = \alpha^\beta(\Theta(t) - x(t)) \tag{45}$$

Of course, it is necessary to define *which* fractional derivative to use in order to verify that a stretched exponential function satisfies this equation.

B. The Rigorous Approach

Another way to approach the problem of finding differential equations that produce stretched exponentials is to use rigorous mathematics. Obviously, this is the ideal way to solve this problem, but this method is also incredibly complicated. Gorska et al. (2017) tackle this problem in this fashion in their paper, "The stretched exponential behavior and its underlying dynamics. The phenomenological approach" [6]. They obtain the equation

$$^{C}\partial_x^\alpha g_\alpha(T^\alpha, x) + \partial_{T^\alpha} g_\alpha(T^\alpha, x) = 0 \tag{46}$$

where $g_\alpha(T^\alpha, x)$ is essentially the Laplace transform of exp($-t^\alpha$) with x as the complex variable of integration, and ${}^C\partial_x^\alpha$ refers to the Caputo fractional derivative:

$$ {}^C\partial_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(y)}{(x-y)^\alpha} dy \tag{47} $$

This is a much more general result than the one derived in the previous section, which holds only for the specific case of a Heaviside step input. It is difficult to see similarities between this equation and equation 45, but, very loosely, we see that both equations prescribe a fractional differential operator as a necessity for producing stretched exponentials. The physical interpretation of this finding is unclear. Equation 44 suggests that the spring like properties of the Maxwell model remain unchanged while the damper needs to be replaced by a device which generates a response proportional to the fractional derivative of position with respect to time. Implementing this in a real control system would be difficult, as there is no known mechanical or electrical device that produces this response. The most likely solution is to approximate fractional derivatives with a combination of purely linear devices.

V. Conclusion

We have explored the Maxwell model in considerable depth, concluding that, while it successfully characterizes the two-regime frequency response of real muscle, it fails to produce stretched exponential functions when subject to a step input. In this regard, it fails to accurately describe the feature of muscle that is believed to contribute to animals' ability to stabilize themselves over long timescales. To address this shortcoming, we then explored modifications to the Maxwell model in order to produce a stretched exponential response. Using a naive mathematical method (i.e. not proved rigorously), we came to the conclusion that a fractional differential operator was necessary. Our findings point to future work in producing approximation methods for fractional differential operators (for applications in the physical world), as well as developing rigorous methods for dealing with fractional differential equations and their solutions.

References

[1] E. Bizzi, W. Chapple, and N. Hogan, "Mechanical properties of muscles: Implications for motor control," *Trends in Neurosciences*, vol. 5, pp. 395–398, 1982.

[2] J. D. Madden, N. A. Vandesteeg, P. A. Anquetil, P. G. Madden, A. Takshi, R. Z. Pytel, S. R. Lafontaine, P. A. Wieringa, and I. W. Hunter, "Artificial muscle technology: physical principles and naval prospects," *IEEE Journal of oceanic engineering*, vol. 29, no. 3, pp. 706–728, 2004.

[3] H. A. Barnes, J. F. Hutton, and K. Walters, *An introduction to rheology*. Elsevier, 1989.

[4] Y.-c. Fung, *Biomechanics: mechanical properties of living tissues*. Springer Science & Business Media, 2013.

[5] K. D. Nguyen, N. Sharma, and M. Venkadesan, "Active viscoelasticity of sarcomeres," *Frontiers in Robotics and AI*, vol. 5, p. 69, 2018.

[6] K. Górska, A. Horzela, K. A. Penson, G. Dattoli, and G. H. Duchamp, "The stretched exponential behavior and its underlying dynamics. the phenomenological approach," *Fractional Calculus and Applied Analysis*, vol. 20, no. 1, pp. 260–283, 2017.