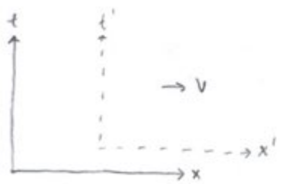
Physics 538

Special Relativity



lorentz transformations relate the positions of an event, as measured in the two coordinate systems

$$x' = (1 - v^2/c^2)^{-1/2}(x - vt)$$

$$ct' = (1 - v^2/c^2)^{-1/2}(ct - \frac{v}{c}x)$$

set c to one. We can do that because c is just a conversion factor between distance and time, and we can arbitrarily scale units.

$$x' = (1 - v^2 \quad)^{-1/2}(x - vt) = \gamma(x - vt)$$

$$t' = (1 - v^2 \quad)^{-1/2}(t - vx) = \gamma(t - vx)$$

$$x = (1 - v^2)^{-1/2}(x + vt) = \gamma(x + vt)$$

$$t = (1 - v^2)^{-1/2}(t + vx) = \gamma(t + vx)$$

lorentz transformation preserves spacetime distance:

$$(\Delta s)^2 = -(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

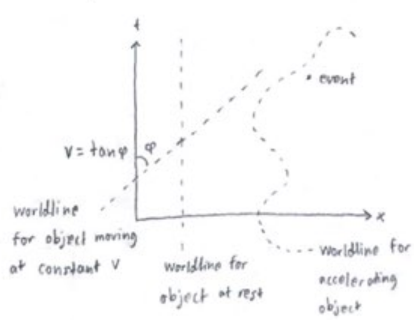
create 4-vector of coordinates:

$$x^\mu = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \qquad \mu = 0, 1, 2, 3$$

$$\Lambda^{\mu'}_{\ \nu} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rows → $\Lambda^{\mu'}{}_{\nu}$ ← columns

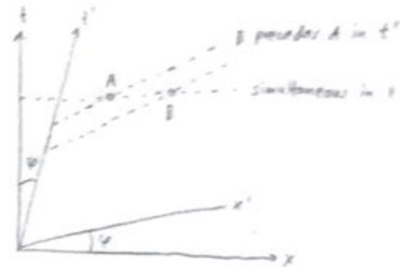
spacetime diagrams:



note, since v cannot exceed $c = 1$, φ cannot exceed 45°

proper lorentz transform:

1. preserves Δs^2
2. preserves sign of t



note: in t', A and B have moved further apart in time, but closer in space

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu$$

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

what's the inverse of $\Lambda^{\mu'}{}_{\nu}$?

$$\Lambda^{\rho}{}_{\mu'} \Lambda^{\mu'}{}_{\nu} = \delta^{\rho}_{\nu}$$ same column because δ is symmetric

$$\frac{\partial x^\rho}{\partial x^\nu} = \delta^\rho_\nu$$

Vectors

imagine particle, on surface of a sphere, that moves on a path parameterized by λ:

$$x^\mu(\lambda) \qquad \mathbb{R} \to \mathbb{R}^4$$

vectors are objects that transform the same way as coordinate differences:

$$\frac{\partial x^\mu}{\partial \lambda}$$

proper time: $(d\tau)^2 = -dt^2 + d\vec{x}^2$

$$V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^\nu$$

convenient basis:

$$\hat{e}_{\langle 0 \rangle} = (1, 0, 0, 0)$$

$$\hat{e}_{\langle 1 \rangle} = (0, 1, 0, 0)$$

$$\hat{e}_{\langle 2 \rangle} = (0, 0, 1, 0)$$

$$\hat{e}_{\langle 3 \rangle} = (0, 0, 0, 1)$$

$e_{\langle \mu \rangle}$ is an element of a basis

$$\underset{\uparrow}{V} = V^{\mu} e_{\langle \mu \rangle}$$

basis independant

how does the basis transform?

$$V^{\nu} e_{\langle \nu \rangle} = \Lambda^{\mu'}{}_{\nu} V^{\nu} e_{\langle \mu' \rangle}$$

$$e_{\langle \nu \rangle} = \Lambda^{\mu'}{}_{\nu} e_{\langle \mu' \rangle}$$

$$\boxed{e_{\langle \mu' \rangle} = \Lambda^{\nu}{}_{\mu'} e_{\langle \nu \rangle}}$$

the basis transforms opposite from how vectors transform

1 - form

linear operation that takes vectors and transforms them into real numbers

$$w(V) \rightarrow \mathbb{R}$$

linear: $w(aV + bU) = a\,w(V) + b\,w(U)$

what basis should we use?

$$w = w_{\mu} \hat{\theta}^{\langle \mu \rangle}$$

$$\hat{\theta}^{\langle \mu \rangle} e_{\langle \nu \rangle} = \delta^{\mu}_{\nu}$$

how do these transform?

$$w(V) = w_{\mu} \hat{\theta}^{\langle \mu \rangle} (V^{\nu} \hat{e}_{\langle \nu \rangle})$$

$$= w_{\mu} V^{\nu} \underbrace{\hat{\theta}^{\langle \mu \rangle} \hat{e}_{\langle \nu \rangle}}_{\delta^{\mu}_{\nu}}$$

$$= w_{\mu} V^{\mu} \quad \text{basis independant}$$

$$w_{\nu} V^{\nu} = w_{\mu'} V^{\mu'}$$

$$w_{\nu} V^{\nu} = w_{\mu'} \Lambda^{\mu'}{}_{\nu} V^{\nu}$$

$$w_{\nu} = w_{\mu'} \Lambda^{\mu'}{}_{\nu}$$

$$\boxed{w_{\mu'} = \Lambda^{\mu}{}_{\mu'} w_{\mu}}$$

ex: consider the 1-form:

$$\frac{\partial f}{\partial x^{\mu}} \equiv \partial_{\mu} f \equiv f_{\mu}$$

$$\partial_{\nu} f = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \partial_{\mu'} f$$

but we know:

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$$

so:

$$\partial_{\nu} f = \Lambda^{\mu'}{}_{\nu} \partial_{\mu'} f$$

since this transforms like a 1-form, it is a 1-form

Tensors

from now on:

Vectors $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ tensors

1-forms $\rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ tensors

an $\begin{bmatrix} n \\ k \end{bmatrix}$ tensor T:

$$T(\underbrace{V_1 \ldots V_k}_{\text{vectors}}, \underbrace{w_1 \ldots w_n}_{\text{1-forms}}) \rightarrow \mathbb{R}$$

it is linear in each argument

use basis:

$$\hat{e}_{\langle \mu_1 \rangle} \otimes \ldots \otimes \hat{e}_{\langle \mu_m \rangle} \otimes \hat{\theta}^{\langle \nu_1 \rangle} \otimes \ldots \otimes \hat{\theta}^{\langle \nu_k \rangle}$$

our indices look like:

$$T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_k}$$

how do tensors transform?

$$T^{\mu_1 \mu_m}{}_{\nu_1 \nu_k} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_m'}{}_{\mu_m} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_k}{}_{\nu_k'} \cdots$$

$$\cdots e_{\langle \mu_1 \rangle} \otimes e_{\langle \mu_m \rangle} \otimes \theta^{\langle \nu_1 \rangle} \otimes \theta^{\langle \nu_k \rangle} T^{\mu_1 \mu_m}{}_{\nu_1 \nu_k}$$

$$T^{\mu_1' \mu_m'}{}_{\nu_1' \nu_k'} = \Lambda^{\mu_1'}{}_{\mu_1} \Lambda^{\mu_m'}{}_{\mu_m} \Lambda^{\nu_1}{}_{\nu_1'} \Lambda^{\nu_k}{}_{\nu_k'} T^{\mu_1 \mu_m}{}_{\nu_1 \nu_k}$$

suppose we have a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ tensor T and a $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ tensor w:

$$T(w, \ldots) = T^{\mu\nu} \hat{e}_{\mu} (w_{\rho} \hat{\theta}^{\rho}) \otimes \hat{e}_{\nu}$$

$$= w_{\rho} \delta^{\rho}_{\mu} T^{\mu\nu} \hat{e}_{\nu}$$

$$= w_{\mu} T^{\mu\nu} \hat{e}_{\nu} \quad \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

this is a contraction with μ

$$T(\ldots, w) = w_{\nu} T^{\mu\nu} \hat{e}_{\mu} \quad \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

this is a contraction with ν

now define force:

$$f^\mu = m \frac{d^2 x^\mu}{d\tau^2}$$

4-momentum: $p^\mu = m U^\mu$

$$u^\mu = (\gamma, \gamma\vec{v}) \underset{\text{small } \vec{v}}{\approx} \left(1 + \frac{\vec{v}^2}{2} \dots, \vec{v} \dots\right)$$

$$p^\mu \approx \Big(\underbrace{m + \tfrac{1}{2}mv^2 \dots}_{\text{relativistic energy}}, \underbrace{m\vec{v} \dots}_{\text{normal momentum}}\Big)$$

p^0 is the energy

in the rest frame:

$$f^\mu = (0, \vec{F})$$

in a moving frame $(\vec{v} \parallel \vec{F})$:

$$f^0 = \gamma \vec{v} \cdot \vec{F}$$

Fluids

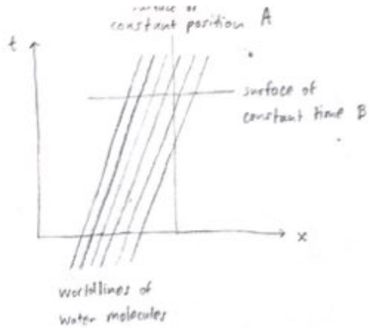
fluids - a collection of particles that we can treat as a continuum

<u>Flux</u>: in the rest frame, we have number density n

$N^\mu = n U^\mu$ is the number flux

ultimately, this will be a useful quantity

* U^μ could be a function of position



number of lines through B is the ordinary flux

number of lines passing through A is density

N^0 is density

$\vec{N}$ is ordinary flux

N^μ is a flux of a particle through a constant x^μ

$$N^\mu = n U^\mu = (n\gamma, n\gamma\vec{v})$$

define something similar:

$T^{\mu\nu}$ = flux of 4-momentum p^μ across surface of constant x^ν

p^0 = energy

p^i = 3-momentum

x^0 = time

x^i = space ... so:

T^{00} = energy density

T^{i0} = momentum density

T^{0i} = energy flux (heat transfer)

T^{ij} = pressure tensor

$T^{\mu\nu}$ is symmetric:

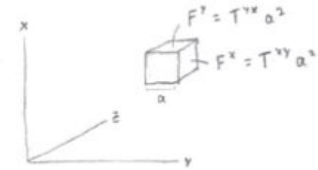
$$T^{\mu\nu} = T^{\nu\mu}$$

because energy and momentum are conserved

$$\partial_\nu T^{\mu\nu} = 0$$

$\partial_\nu T^{0\nu}$ = energy conservation

$\partial_\nu T^{i\nu}$ = momentum conservation



for "perfect fluids" (no heat conduction, no shear pressure):

$$T^{\mu\nu} = \begin{pmatrix} \rho & & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix}$$

in the rest frame:

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\,\eta^{\mu\nu}$$

since: $U^\mu = (1, 0, 0, 0)$ at rest

since it is true in the rest frame, it must be true in all frames

Metric Tensor

the metric is a $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ tensor

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the inverse $\eta^{\mu\nu}$ is clearly the same

if we hadn't set c to 1:

$$\eta_{00} = -c^2 \qquad \eta^{00} = -\frac{1}{c^2}$$

this is invariant under lorentz transform

$$X^{\mu}\eta_{\mu\nu}X^{\nu} = X^{\mu'}\eta_{\mu'\nu'}X^{\nu'}$$

notice: $\quad X^{\mu}\eta_{\mu\nu} = X_{\nu}$

the metric tensor allows us to raise and lower indices

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^{\mu}_{\rho}$$

ex: suppose we have $T^{\mu}{}_{\nu}{}^{\sigma}$ and $S^{\mu}{}_{\mu}{}^{\sigma}$

$$S^{\mu'}{}_{\mu'}{}^{\sigma'} = \underbrace{\Lambda^{\mu'}{}_{\mu}\Lambda^{\sigma'}{}_{\sigma}\Lambda^{\nu}{}_{\mu'}}_{\delta^{\nu}_{\mu}} T^{\mu}{}_{\nu}{}^{\sigma}$$

$$S^{\mu'}{}_{\mu'}{}^{\sigma'} = \delta^{\nu}_{\mu}\Lambda^{\sigma'}{}_{\sigma}T^{\mu}{}_{\nu}{}^{\sigma}$$

$$S^{\mu'}{}_{\mu'}{}^{\sigma'} = \Lambda^{\sigma'}{}_{\sigma}T^{\mu}{}_{\mu}{}^{\sigma}$$

transforms according to the free index because of contraction in μ

due to invariance:

$$\Lambda^{\mu'}{}_{\mu}X^{\mu}\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}X^{\nu} = X^{\mu}\eta_{\mu\nu}X^{\nu}$$

because $\quad X^{\mu'} = \Lambda^{\mu'}{}_{\mu}X^{\mu}$

so it must be:

$$\Lambda^{\mu'}{}_{\mu}\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu} = \eta_{\mu\nu}$$

multiply by $\eta^{\nu\rho}$:

$$\Lambda^{\mu'}{}_{\mu}\underbrace{\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}\eta^{\nu\rho}}_{\text{inverse of } \Lambda^{\mu'}{}_{\mu}} = \eta_{\mu\nu}\eta^{\nu\rho} = \delta^{\rho}_{\mu}$$

$$\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}\eta^{\nu\rho} = \Lambda^{\rho}{}_{\mu'}$$

$$\left(\Lambda_{\mu'}{}^{\rho}\right)^{T} = \Lambda^{\rho}{}_{\mu'}$$

so we get:

$$\left(\Lambda^{\mu'}{}_{\mu}\right)^{-1} = \left(\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}\eta^{\nu\rho}\right)^{T}$$

$$\Lambda^{-1} = \left(\eta\Lambda\eta\right)^{T}$$

symmetries:

$\binom{n}{k}$ tensor $T^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_k}$

$$S^{\alpha_1 \ldots \alpha_n \alpha_{n+1} \ldots \alpha_{n+k}} = \eta^{\alpha_{n+1}\beta_1} \ldots \eta^{\alpha_{n+k}\beta_k} T^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_k}$$

$S^{\mu\nu} = S^{\nu\mu}$ symmetric

$S^{\mu\nu} = -S^{\nu\mu}$ antisymmetric

any tensor can be written as a sum of symmetric and antisymmetric tensors

$$S^{\mu\nu} = \frac{1}{2}\left(S^{\mu\nu} + S^{\nu\mu}\right) + \frac{1}{2}\left(S^{\mu\nu} - S^{\nu\mu}\right)$$

$S^{(123)}$ symmetric

$S^{[123]}$ antisymmetric

4-Velocity

proper time:

$$(d\tau)^2 = -(-dt^2 + d\vec{x}^2) = -\eta_{\mu\nu}dx^{\mu}dx^{\nu}$$

$$d\tau = \sqrt{dt^2 - d\vec{x}^2}$$

$$d\tau = dt\sqrt{1 - \frac{d\vec{x}}{dt}^2}$$

$$d\tau = dt\sqrt{1 - \vec{v}^2}$$

$$\frac{dt}{d\tau} = \gamma$$

$U^{\mu} = \frac{dx^{\mu}}{d\tau}$ "4-velocity"

$$U^{\mu} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \left(\gamma, \frac{d\vec{x}}{dt}\frac{dt}{d\tau}\right)$$

$$U^{\mu} = \gamma(1, \vec{v})$$

$U_{\mu}U^{\mu} = -1$ this object has no free indices and is thus lorentz-invariant

More on $T^{\mu\nu}$

for a perfect fluid:

$$T^{\mu\nu}{}_{,\nu} = \left[(\rho+P)U^\mu U^\nu + P\eta^{\mu\nu}\right]_{,\nu} = 0$$

product rule:

$$(\rho+P)_{,\nu}U^\mu U^\nu + (\rho+P)U^\mu{}_{,\nu}U^\nu +$$
$$(\rho+P)U^\mu U^\nu + P_{,\nu}\eta^{\mu\nu}$$

multiply by U_μ

$$U_\mu T^{\mu\nu}{}_{,\nu} = (\rho+P)_{,\nu}U_\mu U^\mu U^\nu +$$
$$(\rho+P)\left[U_\mu U^\mu{}_{,\nu}U^\nu + U_\mu U^\mu U^\nu{}_{,\nu}\right]$$
$$+ P_{,\nu}U_\mu\eta^{\mu\nu}$$

we know. $U_\mu U^\mu = -1$

$$U_\mu U^\mu{}_{,\nu} = 0$$
$$U_\mu\eta^{\mu\nu} = U^\nu$$

so we get:

$$U_\mu T^{\mu\nu}{}_{,\nu} = -(\rho+P)_{,\nu}U^\nu -$$
$$(\rho+P)U^\nu{}_{,\nu} + P_{,\nu}U^\nu = 0$$

$$\boxed{\left[(\rho+P)U^\nu\right]_{,\nu} = P_{,\nu}U^\nu}$$

this is a conservation statement

EM Stuff

Euler - Lagrange for fields:

$$\partial_\mu\left(\frac{\partial\mathcal{L}(\varphi,\partial_\mu\varphi)}{\partial(\partial_\mu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}$$

for electromagnetism:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu = (-\varphi, \vec{A})$$

write lagrangian using new indices:

$$\mathcal{L} = -\tfrac{1}{4}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) + J^\alpha A_\alpha$$

RHS: $\dfrac{\partial\mathcal{L}}{\partial A_\nu} = J^\alpha\delta^\nu_\alpha = J^\nu$

rewrite:

$$\mathcal{L} = -\tfrac{1}{2}\left(\partial_\alpha A_\beta\partial^\alpha A^\beta - \partial_\alpha A_\beta\partial^\beta A^\alpha\right) + J^\alpha A_\alpha$$

LHS: $\partial_\mu\left(\dfrac{\partial\mathcal{L}(\varphi,\partial_\mu\varphi)}{\partial(\partial_\mu\varphi)}\right) =$

$$-\tfrac{1}{2}\left(\partial^\mu A^\nu + \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu\right)$$

$$= -F^{\mu\nu}$$

putting them together:

$$-\partial_\mu F^{\mu\nu} = J^\nu$$

Curved Space

instead of (x,y), let's use (σ,τ) as our coordinates

$$\sigma(x,y)$$
$$\tau(x,y)$$

$$\Delta\sigma = \frac{\partial\sigma}{\partial x}\Delta x + \frac{\partial\sigma}{\partial y}\Delta y$$

$$\Delta\tau = \frac{\partial\tau}{\partial x}\Delta x + \frac{\partial\tau}{\partial y}\Delta y$$

good coordinates if:

$$\det\begin{pmatrix}\frac{\partial\sigma}{\partial x} & \frac{\partial\sigma}{\partial y}\\ \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y}\end{pmatrix} \neq 0$$

call:

$$\begin{pmatrix}\frac{\partial\sigma}{\partial x} & \frac{\partial\sigma}{\partial y}\\ \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y}\end{pmatrix} = \Lambda^{\alpha'}{}_{\alpha}$$

take polar coordinates, for example:

$$r = \sqrt{x^2+y^2}$$
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\Delta r = \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y$$

$$\Delta\varphi = \frac{-y/x^2}{1+y^2/x^2}\Delta x + \frac{1/x}{1+y^2/x^2}\Delta y$$

$$\Delta r = \cos\varphi\,\Delta x + \sin\varphi\,\Delta y$$

$$\Delta\varphi = -\frac{\sin\varphi}{r}\Delta x + \frac{\cos\varphi}{r}\Delta y$$

$$\Lambda^{\alpha'}{}_{\alpha} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r} \end{pmatrix}$$

our coordinate transformation depends on our position (r, φ)

since our coordinates transform like $\Lambda^{\alpha'}{}_{\alpha}$, vectors do too

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} V^{\alpha}$$

so vectors are attached to points because $\Lambda^{\alpha'}{}_{\alpha}$ is point-dependent

What about our basis vectors?

in cartesian $e_{(\mu)} = (\hat{e}_x, \hat{e}_y)$

$$e_{(\mu')} = \Lambda^{\mu}{}_{\mu'} e_{(\mu)}$$

so our basis vectors in polar are:

$$\Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}$$

$$\Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\left[\begin{array}{l} \hat{e}_r = \cos\varphi\, \hat{e}_x + \sin\varphi\, \hat{e}_y \\ \hat{e}_\varphi = -r\sin\varphi\, \hat{e}_x + r\cos\varphi\, \hat{e}_y \end{array}\right]$$

what should our metric be?

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

$$\hat{e}_r \cdot \hat{e}_r = 1$$

$$\hat{e}_r \cdot \hat{e}_\varphi = 0$$

$$\hat{e}_\varphi \cdot \hat{e}_\varphi = r^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

can also be derived:

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = (dr\cos\varphi - r\sin\varphi\, d\varphi)^2 + (dr\sin\varphi + r\cos\varphi\, d\varphi)^2$$

$$ds^2 = dr^2 + r^2 d\varphi^2$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

what about derivatives of basis vectors?

$$\frac{\partial e_r}{\partial r} = 0 \qquad \frac{\partial e_r}{\partial \varphi} = -\sin\varphi\, e_x + \cos\varphi\, e_y = \frac{e_\varphi}{r}$$

$$\frac{\partial e_\varphi}{\partial r} = \frac{e_\varphi}{r} \qquad \frac{\partial e_\varphi}{\partial \varphi} = -\frac{e_r}{r}$$

derivatives of vectors:

$$\frac{\partial V}{\partial r} = \frac{\partial (V^r e_r + V^\varphi e_\varphi)}{\partial r}$$

$$\frac{\partial V}{\partial r} = \frac{\partial V^r}{\partial r} e_r + V^r \frac{\partial e_r}{\partial r} + \frac{\partial V^\varphi}{\partial r} e_\varphi + V^\varphi \frac{\partial e_\varphi}{\partial r}$$

(product rule)

$$\frac{\partial e_{(\alpha)}}{\partial x^\beta} = e_{(\mu)} \Gamma^{\mu}{}_{\alpha\beta}$$

↑ Christoffel symbols

in our case:

$$\Gamma^{\varphi}{}_{rr} = 0 \qquad \Gamma^{r}{}_{rr} = 0$$

$$\Gamma^{\varphi}{}_{r\varphi} = \frac{1}{r} \qquad \Gamma^{r}{}_{r\varphi} = 0$$

$$\Gamma^{\varphi}{}_{\varphi r} = \frac{1}{r} \qquad \Gamma^{r}{}_{\varphi r} = 0$$

$$\Gamma^{\varphi}{}_{\varphi\varphi} = 0 \qquad \Gamma^{r}{}_{\varphi\varphi} = -r$$

note: christoffel symbols aren't tensors

$\Gamma^{\mu}{}_{\alpha\beta}$ tell us how to take derivatives:

$$\frac{\partial V}{\partial x^\beta} = \frac{\partial (V^\alpha e_{(\alpha)})}{\partial x^\beta}$$

$$= \frac{\partial V^\alpha}{\partial x^\beta} e_{(\alpha)} + V^\alpha e_{(\mu)} \Gamma^{\mu}{}_{\alpha\beta}$$

$$\frac{\partial V}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\gamma \Gamma^{\alpha}{}_{\gamma\beta} \right) e_{(\alpha)}$$

$$\left[\nabla_\beta V^\alpha \equiv \frac{\partial V^\alpha}{\partial x^\beta} + V^\gamma \Gamma^{\alpha}{}_{\gamma\beta} \quad \text{covariant derivative} \right]$$

this satisfies the product rule and is linear... thank you kanye, very cool!

if f is a scalar:

$$f_{,\mu} = f_{;\mu}$$

↑ normal partial derivative ↑ covariant derivative

Summary of covariant derivative:

$$V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + V^{\gamma}\Gamma^{\alpha}_{\gamma\beta} \quad \text{vectors}$$

$$w_{\alpha;\beta} = w_{\alpha,\beta} - w_{\gamma}\Gamma^{\gamma}_{\alpha\beta} \quad \text{one-forms}$$

$$T^{\alpha_1\alpha_2\alpha_3}{}_{;\beta} = T^{\alpha_1\alpha_2\alpha_3}{}_{,\beta} \quad \text{tensors}$$

$$+ T^{\gamma\alpha_2\alpha_3}\Gamma^{\alpha_1}_{\gamma\beta} + T^{\alpha_1\gamma\alpha_3}\Gamma^{\alpha_2}_{\gamma\beta}$$

$$+ T^{\alpha_1\alpha_2\gamma}\Gamma^{\alpha_3}_{\gamma\beta}$$

this all happens because the basis vectors are point-dependant

Γ and $g_{\mu\nu}$

we want $f_{;\alpha\beta} = f_{;\beta\alpha}$

cause scalars

$$\underbrace{(f_{,\alpha})}_{\text{one-form}}{}_{;\beta} = \underbrace{(f_{,\beta})}_{\text{one-form}}{}_{;\alpha}$$

$$f_{,\alpha\beta} - f_{,\gamma}\Gamma^{\gamma}_{\alpha\beta} = f_{,\beta\alpha} - f_{,\gamma}\Gamma^{\gamma}_{\beta\alpha}$$

$$\left[\text{so: } \Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha} \quad (\text{symmetric})\right]$$

we also want: $g_{\alpha\gamma}\left(V^{\alpha}{}_{;\beta}\right) = \left(g_{\alpha\gamma}V^{\alpha}\right)_{;\beta}$

$$g_{\alpha\gamma}\left(V^{\alpha}{}_{,\beta}\right) = g_{\alpha\gamma;\beta}V^{\alpha} + g_{\alpha\gamma}\left(V^{\alpha}{}_{;\beta}\right)$$

so it should be the case that: $\left[g_{\alpha\gamma;\beta} = 0\right]$

$$g_{\alpha\gamma;\beta} = g_{\alpha\gamma,\beta} - g_{\delta\gamma}\Gamma^{\delta}_{\alpha\beta} - g_{\alpha\delta}\Gamma^{\delta}_{\gamma\beta} = 0$$

plus the two other equations gotten from cyclic permutations of $\alpha\gamma\beta$

combining these equations:

$$g_{\beta\alpha,\gamma} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} - 2g_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta} = 0$$

$$\left[\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}\left(g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma}\right)\right]$$

Covarient derivative

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + V^{\rho}\Gamma^{\nu}_{\rho\mu}$$

a. derivative operator (linear, product rule)

b. turns $\binom{n}{k}$ tensors into $\binom{n}{k+1}$ tensors

c. compatible with raising and lowering indices

$$\nabla_{\mu'}V^{\nu'} = \Lambda^{\nu'}{}_{\nu}\Lambda^{\mu}{}_{\mu'}\nabla_{\mu}V^{\nu}$$

$$\nabla_{\mu'}V^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\nabla_{\mu}V^{\nu}$$

$$\nabla_{\mu'}V^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\partial_{\mu}V^{\nu} + V^{\alpha}\Gamma^{\nu}_{\alpha\mu}\right)$$

$$\partial_{\mu'}V^{\nu'} + V^{\alpha'}\Gamma^{\nu'}_{\alpha'\mu'} \rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}V^{\nu}\right) + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}V^{\alpha}\Gamma^{\nu'}_{\alpha'\mu'}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}}V^{\nu} + \frac{\partial x^{\nu'}}{\partial x^{\nu}}\partial_{\mu}V^{\nu}\right) + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}V^{\alpha}\Gamma^{\nu'}_{\alpha'\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\left(\partial_{\mu}V^{\nu} + V^{\alpha}\Gamma^{\nu}_{\alpha\mu}\right)$$

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}V^{\alpha}\Gamma^{\nu'}_{\alpha'\mu'} =$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}V^{\alpha}\Gamma^{\nu}_{\alpha\mu} -$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\alpha}}V^{\alpha}$$

everything has V^{α}, so we can get rid of it

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\Gamma^{\nu'}_{\alpha'\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\alpha\mu}$$

$$- \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\alpha}}$$

$$\boxed{\Gamma^{\nu'}_{\rho'\mu'} = \frac{\partial x^{\alpha}}{\partial x^{\rho'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\alpha\mu} - \frac{\partial x^{\alpha}}{\partial x^{\rho'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\alpha}}}$$

so Γ transforms like a tensor except for the inhomogeneous term

Applications

$$\nabla_\mu V^\mu = \nabla_\mu \overset{\text{scalar}}{f}{}^{,\mu}$$

$$= \partial_\mu f^{,\mu} + f^{,\rho}\,\Gamma^{\mu}_{\rho\mu}$$

we know:

$$\Gamma^{\mu}_{\mu\alpha} = \frac{1}{\sqrt{|g|}}\,\partial_\alpha \sqrt{|g|}$$

$$\nabla_\mu f^{,\mu} = \partial_\mu f^{,\mu} + f^{,\mu}\frac{1}{\sqrt{|g|}}\,\partial_\mu \sqrt{|g|}$$

$$= \frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}\, f^{,\mu}\right)$$

$$\boxed{\nabla_\mu f^{,\mu} = \frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}\; g^{\mu\nu} f_{,\nu}\right)}$$

in spherical coordinates:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

we want to know the ∇^2 operator:

$$\nabla^2 f = \nabla_\mu f^{,\mu}$$

$$|g| = r^4 \sin^2\theta$$

$$\sqrt{|g|} = r^2 \sin\theta$$

$$\nabla^2 f^{,\mu} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

The Geodesic

parallel transport of vectors along closed paths in curved space doesn't always return the vector to the same place

$$x^\mu(\lambda)$$

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu f$$

but ∂_μ doesn't change properly in curved space, so let's use the covariant derivative instead

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu$$

in flat space, our definition of a straight line is:

$$\frac{d}{d\lambda}\left(\frac{dx^\nu}{d\lambda}\right) = 0$$

so in curved space, straight line:

$$\frac{D}{\partial\lambda}\frac{dx^\mu}{d\lambda} = 0$$

$$\frac{dx^\nu}{d\lambda}\left(\nabla_\nu \frac{dx^\mu}{d\lambda}\right) = 0$$

$$\frac{dx^\nu}{d\lambda}\left(\partial_\nu \frac{dx^\mu}{d\lambda} + \Gamma^{\mu}_{\rho\nu}\frac{dx^\rho}{d\lambda}\right) = 0$$

$$\frac{dx^\nu}{d\lambda}\partial_\nu = \frac{dx^\nu}{d\lambda}\frac{d}{dx^\nu} = \frac{d}{d\lambda}$$

$$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = 0}$$

this is the geodesic equation

solutions to this equation for a given $g_{\mu\nu}$ (which determines Γ) form straight lines — lines that free falling objects take — in $g_{\mu\nu}$

in flat space, $\Gamma = 0$ and we recover:

$$\frac{d^2x^\mu}{d\lambda^2} = 0$$

where $x^\mu(\lambda)$ is the path that a non-accelerating object follows

example:

assume x^μ is on a geodesic, let's look at a nearby geodesic

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

"function of"

$$\frac{d^2}{d\tau^2}\left(x^\mu + \varepsilon^\mu\right) + \Gamma^{\mu}_{\alpha\beta}\left(x^\nu + \varepsilon^\nu\right)\Big[\frac{d}{d\tau}\left(x^\alpha + \varepsilon^\alpha\right)\frac{d}{d\tau}\left(x^\beta + \varepsilon^\beta\right)\Big] = 0$$

expand assuming ε^μ is small:

$$\frac{d^2\varepsilon^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta,\nu}\,\varepsilon^\nu \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} +$$

$$2\,\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\varepsilon^\beta}{d\tau} = 0$$

remember: $\frac{D}{D\tau} = \frac{dx^\rho}{d\tau}\nabla_\rho$

we want to know $\frac{D^2\varepsilon^\mu}{D\tau^2}$ because that tells us the path of the nearby object

$$\frac{D^2\varepsilon}{D\tau^2} = \frac{dx^\alpha}{d\tau}\nabla_\alpha\left(\frac{dx^\beta}{d\tau}\nabla_\beta\,\varepsilon^\mu\right)$$

$$= \frac{dx^\alpha}{d\tau}\nabla_\alpha\left[\underbrace{\frac{dx^\beta}{d\tau}\left(\frac{\partial\varepsilon^\mu}{\partial x^\beta} + \Gamma^\mu_{\delta\beta}\,\varepsilon^\delta\right)}_{\text{this is a vector}}\right]$$

$$= \frac{dx^\alpha}{d\tau}\partial_\alpha\left[\frac{dx^\beta}{d\tau}\left(\underbrace{\frac{\partial\varepsilon^\mu}{\partial x^\beta}}_{\frac{d\varepsilon^\mu}{d\tau}} + \Gamma^\mu_{\delta\beta}\,\varepsilon^\delta\right)\right] +$$

$$\frac{dx^\alpha}{d\tau}\Gamma^\mu_{\gamma\alpha}\frac{dx^\beta}{d\tau}\left(\frac{\partial\varepsilon^\gamma}{\partial x^\beta} + \Gamma^\gamma_{\delta\beta}\,\varepsilon^\delta\right)$$

now take derivatives for each term:

$$= \underbrace{\frac{d^2\varepsilon^\mu}{d\tau^2}}_{\text{one}} + \underbrace{\frac{dx^\alpha}{d\tau}\partial_\alpha\left(\frac{dx^\beta}{d\tau}\Gamma^\mu_{\delta\beta}\,\varepsilon^\delta\right)}_{\text{two}} +$$

$$\Gamma^\mu_{\gamma\alpha}\frac{dx^\alpha}{d\tau}\frac{d\varepsilon^\gamma}{d\tau} + \Gamma^\mu_{\gamma\alpha}\Gamma^\gamma_{\delta\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta$$

note, from the equation at top of page:

$$\frac{d^2\varepsilon^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta,\nu}\,\varepsilon^\nu\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} -$$

$$2\,\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\varepsilon^\beta}{d\tau}$$

term two:

$$\frac{dx^\alpha}{d\tau}\partial_\alpha\left(\frac{dx^\beta}{d\tau}\Gamma^\mu_{\delta\beta}\,\varepsilon^\delta\right) = \frac{d^2x^\beta}{d\tau}\Gamma^\mu_{\delta\beta}\,\varepsilon^\delta$$

$$+ \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\Gamma^\mu_{\delta\beta,\alpha}\,\varepsilon^\delta + \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\Gamma^\mu_{\delta\beta}\frac{d\varepsilon^\delta}{dx^\alpha}$$

we can replace $\frac{d^2x^\beta}{d\tau}$ with $\Gamma^\beta_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau}$ since it follows the geodesic

after combining the expanded terms:

$$= -\Gamma^\mu_{\alpha\beta,\delta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta - \Gamma^\beta_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau}\Gamma^\mu_{\beta\delta}\,\varepsilon^\delta$$

$$+ \Gamma^\mu_{\delta\beta,\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta + \Gamma^\mu_{\gamma\alpha}\Gamma^\gamma_{\delta\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta$$

$$\frac{D^2\varepsilon^\mu}{D\tau^2} = \left(\Gamma^\mu_{\delta\beta,\alpha} - \Gamma^\mu_{\alpha\beta,\delta} + \Gamma^\mu_{\lambda\alpha}\Gamma^\lambda_{\delta\beta} - \Gamma^\mu_{\lambda\delta}\Gamma^\lambda_{\alpha\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta$$

$$\frac{D^2\varepsilon^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\delta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\varepsilon^\delta$$

$$\boxed{R^\mu{}_{\alpha\beta\delta} = \Gamma^\mu_{\beta\delta,\alpha} - \Gamma^\mu_{\alpha\beta,\delta} + \Gamma^\mu_{\lambda\alpha}\Gamma^\lambda_{\delta\beta} - \Gamma^\mu_{\lambda\delta}\Gamma^\lambda_{\alpha\beta}}$$

this is the Reimmann curvature tensor. It describes deviations from flat space. The metric alone doesn't tell us whether we are in flat space or not:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix} \leftrightarrow \begin{pmatrix}\sigma^2+\tau^2 & 0\\ 0 & \sigma^2+\tau^2\end{pmatrix}$$

Euclidean Polar Parabolic

all these metrics describe flat space, just with different coordinates

Killing Equation

$$x'^\mu = x^\mu + \varepsilon K^\mu(x)$$

$$g_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}g'_{\alpha\beta}(x')$$

space (the metric) is unchanged under killing vector transformations which are isometries

expand in powers of ε:

$$g_{\mu\nu}(x) = \left(\delta^\alpha_\mu + \varepsilon K^\alpha{}_{,\mu}\right)\left(\delta^\beta_\nu + \varepsilon K^\beta{}_{,\nu}\right)\left(g_{\alpha\beta} + \varepsilon K^\rho g_{\alpha\beta,\rho}\right)$$

take linear terms:

$$g_{\mu\nu}(x) = g_{\mu\nu}(x) + \varepsilon\left(K^\alpha{}_{,\mu}\,g_{\alpha\nu} + K^\beta{}_{,\nu}\,g_{\mu\beta} + K^\rho g_{\mu\nu,\rho}\right) + O(\varepsilon^2)$$

$$0 = \left(K^\alpha g_{\alpha\nu}\right)_{,\mu} - K^\alpha g_{\alpha\nu,\mu} + \left(K^\beta g_{\mu\beta}\right)_{,\nu} - K^\beta g_{\mu\beta,\nu} + K^\rho g_{\mu\nu,\rho}$$

$$0 = K_{\nu,\mu} + K_{\mu,\nu} - K^\rho\left(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}\right)$$

$$0 = K_{\nu,\mu} + K_{\mu,\nu} - \underbrace{K^\sigma g_{\tau\sigma}g^{\tau\rho}\left(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}\right)}_{\text{this looks like the connection coefficients}}$$

$$0 = K_{\nu,\mu} + K_{\mu,\nu} - K_\tau\,2\,\Gamma^\tau_{\mu\nu}$$

this looks like a covarient derivative

$$\boxed{0 = K_{\nu;\mu} + K_{\mu;\nu}}$$

solutions to this equation represent isometries in space

$$\frac{D}{D\lambda}\left(\underbrace{K_\mu\frac{dx^\mu}{d\lambda}}_{\text{conserved along geodesics}}\right) = 0$$

Equivalence Principle

this says that the inertial mass and the gravitational mass are the same

Formally, we have:

In small regions (locally) there are inertial coordinate systems in which all laws of physics are those of special relativity

i.e. commas go to semi-colons

consider coordinate transformations:

$$g_{\mu'\nu'} = \underbrace{\frac{\partial x^{\alpha}}{\partial x^{\mu'}}}_{\Lambda^{\alpha}{}_{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} g_{\alpha\beta}$$

taylor expand $g_{\alpha\beta}$ in the neighborhood of x_0:

$$g_{\alpha\beta} = g_{\alpha\beta}(x_0) + (x-x_0)^{\gamma} g_{\alpha\beta,\gamma}\Big|_{x_0}$$

$$+ \frac{1}{2}(x-x_0)^{\gamma}(x-x_0)^{\delta} g_{\alpha\beta,\gamma\delta}\Big|_{x_0} \cdots$$

do the same thing for the transformation:

$$\Lambda^{\alpha}{}_{\mu'} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\Big|_{x_0} + (x-x_0)^{\gamma'} \frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\mu'}}\Big|_{x_0}$$

$$+ \frac{1}{2}(x-x_0)^{\gamma'}(x-x_0)^{\delta'} \frac{\partial^3 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\delta'}\partial x^{\mu'}}\Big|_{x_0} \cdots$$

now let's count the number of free components in these objects

object	# comp. in d-dimensions	4-d
$g_{\mu\nu}$	$\frac{d(d+1)}{2}$	10
$g_{\mu\nu,\alpha}$	$\frac{d^2(d+1)}{2}$	40
$g_{\mu\nu,\alpha\beta}$	$\frac{d^2(d+1)^2}{4}$	100
$\frac{\partial x^{\alpha}}{\partial x^{\mu'}}$	d^2	16
$\frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}}$	$\frac{d^2(d+1)}{2}$	40
$\frac{\partial^3 x^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}\partial x^{\gamma'}}$	$\frac{d^2(d+1)(d+2)}{6}$	80

moral of the story: we always have enough free parameters to:

1. at x_0, make $g_{\mu\nu} \to \eta_{\mu\nu}$

   (with 6 degrees of freedom to spare, the 6 transformations in the lorentz group)

2. at x_0, make all $g_{\mu\nu,\alpha} = 0$

   because we have 40 d.o.f. and 40 possible transformations

   $\Rightarrow \Gamma(x_0) = 0$

3. have $\frac{1}{4}d^2(d+1)^2 - \frac{1}{6}d^2(d+1)(d+2)$

   $= \frac{1}{12}d^2(d-1)$ spare second-derivatives of $g_{\mu\nu}$: this is the number of components in $R^{\alpha}{}_{\beta\gamma\delta}$

   in 4-d: $100 - 80 = 20$ components

Curvature

commutator of 2 covariant derivatives:

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = \nabla_{\mu}\nabla_{\nu}V^{\rho} - \nabla_{\nu}\nabla_{\mu}V^{\rho}$$

$$= \partial_{\mu}(\nabla_{\nu}V^{\rho}) - \Gamma^{\sigma}_{\nu\mu}(\nabla_{\sigma}V^{\rho}) + \Gamma^{\rho}_{\sigma\mu}(\nabla_{\nu}V^{\sigma})$$

$$= \partial_{\mu}(\partial_{\nu}V^{\rho} + \Gamma^{\rho}_{\sigma\nu}V^{\sigma}) - \Gamma^{\sigma}_{\nu\mu}(\partial_{\sigma}V^{\rho} + \Gamma^{\rho}_{\varsigma\sigma}V^{\varsigma}) + \Gamma^{\rho}_{\sigma\mu}(\partial_{\nu}V^{\sigma} + \Gamma^{\sigma}_{\varsigma\nu}V^{\varsigma})$$

$$= \partial_{\mu}\partial_{\nu}V^{\rho} + \Gamma^{\rho}_{\sigma\nu,\mu}V^{\sigma} + \Gamma^{\rho}_{\sigma\nu}V^{\sigma}{}_{,\mu} - \Gamma^{\sigma}_{\nu\mu}V^{\rho}{}_{,\sigma} - \Gamma^{\sigma}_{\nu\mu}\Gamma^{\rho}_{\varsigma\sigma}V^{\varsigma} + \Gamma^{\rho}_{\sigma\mu}V^{\sigma}{}_{,\nu}$$

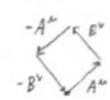
$$+ \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\varsigma\nu}V^{\varsigma}$$

this is the first term in the commutator. Add to each term the opposite (opposite sign, μ and ν switched)

after some cancellations, we get:

$$\boxed{[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma}}$$

$[\nabla_{\mu}, \nabla_{\nu}]$ describes parallel transport around a closed loop



equation for parallel transport:

$$\frac{dx^{\mu}}{d\lambda}\nabla_{\mu}V^{\rho} = 0$$

$$\frac{d}{d\lambda}V^{\rho} + \frac{dx^{\mu}}{d\lambda}\Gamma^{\rho}_{\sigma\mu}V^{\sigma} = 0$$

if the vector comes back to itself, we are in flat space and $R = 0$

Properties of R

1. it's a tensor

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\eta\mu\nu} V^\eta$$

$$R^\rho_{\eta\mu\nu} = \Gamma^\rho_{\eta\nu,\mu} - \Gamma^\rho_{\eta\mu,\nu} + \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\eta\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\eta\mu}$$

2. it's antisymmetric under exchange of the last two indices

$$R^\rho{}_{\eta\mu\nu} = -R^\rho{}_{\eta\nu\mu}$$

3. assume R is locally minkowski

i.e. $\Gamma = 0$ but not $\Gamma_{,\mu}$

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^\rho{}_{\beta\gamma\delta}$$

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho}\left(\Gamma^\rho_{\beta\delta,\gamma} - \Gamma^\rho_{\beta\gamma,\delta}\right)$$

doing some messy math:

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\big(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma}\big)$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0$$

4. number of free components in d dimensions

$R_{[\alpha\beta][\gamma\delta]}$ [ ] antisymmetric, ↻ symmetric

[ ] gives us $\frac{d(d-1)}{2}$

↻ gives us $\frac{d(d-1)}{2} + 1$

additionally, we have:

$$R_{[\alpha\beta\gamma]\delta} \rightarrow \frac{d(d-1)(d-2)(d-3)}{24}$$

combining these, we get:

$$\boxed{\#\text{ free comps.} = \frac{d^2(d^2-1)}{12}}$$

5. $\nabla_\eta R_{\alpha\beta\gamma\delta} + \nabla_\alpha R_{\beta\eta\gamma\delta} + \nabla_\beta R_{\eta\alpha\gamma\delta} = 0$

this is called the Bianchi identity

6. related quantities

$$R_{\alpha\beta} = R^\lambda{}_{\alpha\lambda\beta} \quad \text{(Ricci tensor)}$$

$$R = g^{\alpha\beta} R_{\alpha\beta} \quad \text{(curvature scalar)}$$

$$\nabla^\beta R_{\alpha\beta} - \tfrac{1}{2}\nabla_\alpha R = 0 \quad \text{(also Bianchi identity)}$$

Einstein's Equations

in classical mechanics:

$$\vec{a} = -\nabla\varphi$$

↑ acceleration   some potential

$$\nabla^2\varphi = 4\pi G_N \rho \quad \text{— mass density}$$

in 4-vector form:

$$\rho \rightarrow T_{\mu\nu}$$

$$\nabla^2\varphi \rightarrow \text{"}[\nabla^2 g]_{\mu\nu}\text{"}$$

$T_{\mu\nu}$ is covariently conserved, so " $\frac{\cdot}{\mu\nu}$ must also be covariently conserved

we know:

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is conserved:

bianchi identity: $\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0$

$$\nabla_\nu = g_{\mu\nu}\nabla^\mu$$

$$\nabla^\mu R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\nabla^\mu R = 0$$

$$\nabla^\mu\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) = 0$$

this is the definition of covarient conservation

also, $g_{\mu\nu}$ is conserved

putting it all together:

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

Λ is the cosmological constant

$\Lambda g_{\mu\nu}$ represents the vacuum energy:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

for a perfect fluid:

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$$

$$p = -\rho$$

$$\rho = \frac{\Lambda}{8\pi G}$$ the energy density in a vaccuum

Lagrange Formulation

volume element in n dimensions:

$$d^n x' = \underbrace{\det\left(\frac{dx'}{dx}\right)}_{\text{Jacobian}} d^n x \qquad *$$

we want a coordinate independant volume element

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{dx^{\nu}} g_{\mu'\nu'}$$

$$|g| = J^2 |g'|$$

consider the volume element $\sqrt{-|g|}\, d^n x$

from $*$:

$$\sqrt{-|g'|}\, d^n x' = \sqrt{-|g'|}\; J\, d^n x$$

$$\sqrt{-|g'|} = \frac{1}{J} \sqrt{-|g|}$$

$$\sqrt{-|g|}\, d^n x' = \sqrt{-|g|}\, d^n x \quad \checkmark$$

so $\sqrt{-|g|}\, d^n x$ is our invariant volume element

now consider the action:

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{-|g|} \left(-2\Lambda + R\right)$$

the Euler-Lagrange equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

now let's add squared terms to the action:

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{-|g|} \left[(-2\Lambda + R) + \beta\left(R^2 + R_{\mu\nu} R^{\mu\nu}\right)\right]$$

can we neglect higher order terms?

β must have units of m^2 to offset the fact that the terms are squared

$$\beta = \frac{G\hbar}{c^3} \sim 10^{-70}$$

so yeah, we can ignore them

this was for empty space. We can add the lagrangian for matter:

$$\mathcal{L}_m = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi\, \partial_\nu \varphi - V(\varphi)$$

$$\frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{-|g|}\, \mathcal{L}_m\right) = T_{\mu\nu}$$

setting the two euler-lagrange equations equal, and adding some constants of proportionality:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Shwarzchild

we are looking for isotropic, time independant solutions

so we can write:

$$g_{\mu\nu} = \left(\begin{array}{c|c} g_{00} & g_{0i} \\ \hline g_{i0} & g_{ij} \end{array}\right)$$

$$t, \vec{x} \quad \text{s.t.} \quad \sqrt{\vec{x}\cdot\vec{x}} = r \qquad \frac{\vec{x}\cdot d\vec{x}}{\sqrt{\vec{x}\cdot\vec{x}}} = dr$$

$$g_{00} = -f(r)$$

$$g_{0i} = E(r)\, x_i$$

$$g_{ij} = D(r)\, x_i x_j + C(r)\, \delta_{ij}$$

$$ds^2 = -f(r)\, dt^2 + 2E(r)\, \vec{x}\cdot d\vec{x}\, dt + D(r)(\vec{x}\cdot d\vec{x})^2 + C(r)(d\vec{x})^2$$

the Schwarzchild solution looks for solutions to:

$$R_{\mu\nu} = 0$$

we can simplify ds^2 by noticing that $\vec{x}\cdot d\vec{x} = r\,dr$

consider the transformations:

$$t \to t + \Phi(r)$$

$$dt \to dt + \Phi' dr$$

$$-f(r)\, dt^2 \to -f(r)\left(dt^2 + \underline{2\Phi'(r)\, dt} + \Phi'^2 dr^2\right)$$

$$2E(r)\, dr\, dt \to 2E(r)\, r \left(\underline{dt\, dr} + \Phi' dr^2\right)$$

we can choose $\Phi(r)$ s.t. the two underlined terms cancel

$$-f(r)\,\Phi' + E(r)\, r = 0$$

this transformation makes the off-diagonal elements go away

now make another transformation:

$$r^2 C(r) \to r^2$$

also, write: $(d\hat{x})^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$

our line element becomes:

$$ds^2 = -f(r)\,dt^2 - f(r) I'^2 dr^2 + 2E(r) r I'^2 dr^2 + D(r) r^2 dr^2 + C(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)$$

$$ds^2 = A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

$$\left[ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \right]$$

this is the most general isotropic, time-independant metric, and the one we'll use to look for solutions to $R_{\mu\nu} = 0$

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - (\partial_r \alpha)(\partial_r \beta) + \frac{2}{r}\partial_r \alpha\right]$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + (\partial_r \alpha)(\partial_r \beta) + \frac{2}{r}\partial_r \alpha$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r \beta - \partial_r \alpha) - 1\right] + 1$$

$$R_{\varphi\varphi} = \sin^2\theta \, R_{\theta\theta}$$

these can be derived from the connections, which can be derived from the metric

$$R_{tt} e^{-2(\alpha-\beta)} + R_{rr} = \frac{2}{r}\partial_r(\alpha+\beta)$$

since we want $R_{\mu\nu} = 0$:

$$\frac{2}{r}\partial_r(\alpha+\beta) = 0$$

$$\alpha = -\beta$$

$$R_{\theta\theta} = e^{2\alpha}(-2r\,\partial_r \alpha - 1) + 1 = 0$$

$$e^{2\alpha}(-2r\,\partial_r \alpha - 1) = 1$$

$$\partial_r(r e^{2\alpha}) = 1$$

$$r e^{2\alpha} = r + \text{const}$$

call the constant R_s, the schwarzchild radius

$$e^{2\alpha} = 1 - \frac{R_s}{r}$$

$$e^{2\beta} = \left(1 - \frac{R_s}{r}\right)^{-1}$$

$$\boxed{ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\Omega^2}$$

this equation is valid everywhere outside the matter, i.e. $T_{\mu\nu} = 0$

taking the classical limit, we get:

$$R_s = 2MG$$

note: the signature of the time coordinate changes sign at $r = R_s$

Implications

the schwarzchild solution is valid outside a spherically symmetric massive object

we previously looked at isometries that do not change the metric

$$x^\mu \to x^\mu + \varepsilon K^\mu(x)$$

K^μ satisfy $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$

there are 2 obvious symmetries:

$$t \to t + \varepsilon$$

$$\varphi \to \varphi + \varepsilon$$

$$K_{(t)}^{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad K_{(\varphi)}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{bmatrix} t \\ r \\ \theta \\ \varphi \end{bmatrix}$$

additionally, we have:

$$K_\mu \frac{dx^\mu}{d\lambda} \text{ is conserved}$$

$$K_{(t)\mu} = \left(-\left(1 - \frac{R_s}{r}\right), 0, 0, 0\right)$$

$$K_{(\varphi)\mu} = \left(0, 0, 0, r^2 \sin^2\theta\right)$$

for simplicity, say $\theta = \frac{\pi}{2}$

$$K_{(\varphi)\mu} = (0, 0, 0, r^2)$$

our conserved quantities are:

$$\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\lambda} = E$$

$$r^2 \frac{d\varphi}{d\lambda} = L$$

now, let's say:

$$-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \varepsilon \begin{cases} 1 & \text{massive particles} \\ 0 & \text{massless particles} \end{cases}$$

$$\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{1}{1-\frac{R_s}{r}}\right)\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = \varepsilon$$

plugging in our conserved quantities:

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right) = E^2}$$

this is the radial equation of motion for a particle in the schwarzchild field

what is the time it takes for a freely falling particle to reach R_s?

$$\Delta\tau = -\int_{r_0}^{R_s}\frac{d\tau}{dr}\,dr = \int_{r_0}^{R_s}\frac{1}{\sqrt{E^2-\left(1-\frac{R_s}{r}\right)}}\,dr$$

rescaling variables, we get:

$$\Delta\tau = R_s\int_{t_0}^{1}\frac{1}{t^2\sqrt{t-t_0}}\,dt$$

bottom line: $\Delta\tau$ is finite

what about coordinate time?

$$\Delta t = -\int_{r_0}^{R_s}\frac{dt}{dr}\,dr = -\int_{r_0}^{R_s}\frac{\frac{dt}{d\tau}}{\frac{dr}{d\tau}}\,dr$$

$$\left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau} = E$$

$$\frac{dr}{d\tau} = \sqrt{E^2-1+\frac{R_s}{r}}$$

$$\Delta t = \int_{r_0}^{R_s}\frac{E}{\left(1-\frac{R_s}{r}\right)\sqrt{\frac{R_s}{r}-\frac{R_s}{r_0}}}\,dr$$

this integral goes to infinity, meaning if you throw something into a black hole, it will take infinite time for it to reach R_s in your time, but finite time from the perspective of the object

Classical tests of GR:

a) gravitational redshift

b) precession of perihelia

c) bending of light

d) radar echo delay

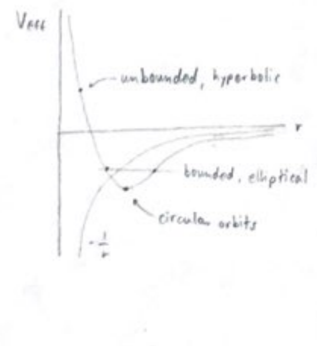
e) binary pulsar

Orbits

classically:

$$\frac{1}{2}\dot{r}^2 + \underbrace{\left(V(r)+\frac{L^2}{2r^2}\right)}_{\text{effective potential}} = E$$

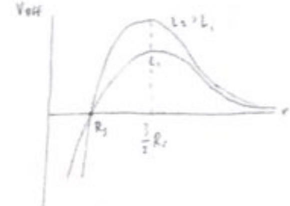
in newtonian mechanics: $V(r) = \frac{1}{r}$



according to the boxed equation:

in the massless case $\varepsilon = 0$:

$$V_{eff} = \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}\right)$$

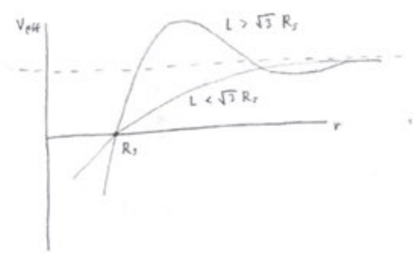


$$V_{eff\,(max)}\left(\frac{3}{2}R_s\right) = \left(1-\frac{2}{3}\right)\left(\frac{4}{9}\frac{L^2}{R_s^2}\right)$$

$$V_{eff\,(max)} = \frac{4}{27}\frac{L^2}{R_s^2}$$

in the massive case:

$$V_{eff} = \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right)$$



if $L < \sqrt{3}\,R_s$, the particle will always be sucked into the black hole

if $L > \sqrt{3}\,R_s$ and is energetic enough to get close to R_s, it will fall in. If it isn't energetic enough it will obey one of the three classical orbits

Perihelia Precession

for massive bound orbits, make the substitution:

$$\frac{1}{r} = \frac{R_s^2}{2L^2} u$$

then we get:

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 - 2u - u^3 \frac{R_s^2}{2L^2} = \frac{4L^2(E^2-1)}{R_s^2}$$

take a φ derivative:

$$2\frac{du}{d\varphi}\left(\frac{d^2u}{d\varphi^2}\right) + 2u\frac{du}{d\varphi} - 3u^2\frac{R_s^2}{2L^2}\frac{du}{d\varphi} - 2\frac{du}{d\varphi} = 0$$

$$\frac{d^2u}{d\varphi^2} + u - 1 - \frac{3}{2}u^2\frac{R_s^2}{2L^2} = 0$$

$$R_s = 2MG$$

$$\frac{d^2u}{d\varphi^2} + u - 1 = \frac{3M^2G^2}{L^2}u^2$$

standard Keplerian orbit

treat the RHS as a perturbation:

call $\frac{3M^2G^2}{L^2} = \zeta$

$$u = u_0 + \zeta u_1 + \zeta^2 u_2 \ldots$$

Keep only first-order terms in ζ

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \zeta u_0^2$$

$u_0 = 1 + e\cos\varphi$ (the equation for an unperturbed Keplerian orbit)

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \zeta(1 + e\cos\varphi)^2$$

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \zeta(1 + 2e\cos\varphi + e^2\cos^2\varphi)$$

the solution to this is:

$$\boxed{u_1 = \zeta\left(1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6}\cos(2\varphi)\right)}$$

thus, the perturbed orbit is not closed due to the fact that we have non-periodic terms in u_1

the maxima of u are the perihelia of the orbit:

$$\frac{du}{d\varphi} = -e\sin\varphi + \zeta\left(e\sin\varphi + e\varphi\cos\varphi + \frac{e^2}{3}\sin(2\varphi)\right) = 0$$

$$0 = -e\sin\delta + \zeta(2\pi e + \text{small})$$

$\delta = 2\pi\zeta$ where δ is the angle between one perihelion and the next

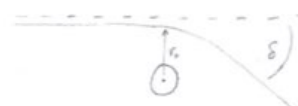
substituting in for ζ:

$$\boxed{\delta = \frac{6\pi GM}{(1-e^2)a}}$$

e is the eccentricity of the orbit and a is the major radius of the orbit

Light Bending

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{R_s}{r}\right)} + r^2 d\Omega^2$$



for non-massive objects (light):

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}\right) = E^2$$

$$\frac{dr}{d\varphi} = \frac{dr}{d\lambda}\frac{d\lambda}{d\varphi} = \frac{r^2}{L}\frac{dr}{d\lambda}$$

$$\frac{dr}{d\varphi} = \frac{r^2}{L}\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right)}$$

$$\frac{d\varphi}{dr} = \frac{L}{r^2}\left(E^2 - \frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right)\right)^{-1/2}$$

$$\Delta\varphi = 2\int_{r_0}^{\infty}\frac{L}{r^2}\left[E^2 - \frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right)\right]^{-1/2}dr$$

$$u = \frac{1}{r}$$

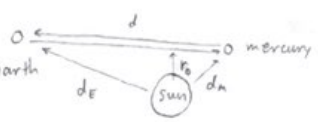
$$\Delta\varphi = 2\int_0^{u_0}\left[u_0^2 - u^2 + R_s(u^3 - u_0^3)\right]^{-1/2}du$$

to linear order:

$$\Delta\varphi = \pi + \frac{2R_s}{r_0}$$

$$\boxed{\Delta\varphi = \pi + \frac{4MG}{r_0}}$$

Shapiro Time Delay



bounce a radio signal off mercury, passing close to the sun: it takes longer than expected

end result:

$$t = \sqrt{d^2 - b^2} + R_s \ln\left[\frac{d + \sqrt{d^2 - b^2}}{b}\right]$$

$$+ \frac{R_s}{2} \frac{\sqrt{d^2 - b^2}}{d + b}$$

calculation:

we want $\frac{dt}{dr}$: we have:

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{R_s}{r}\right)\frac{L^2}{r^2} = E^2$$

$$\frac{dr}{d\lambda} = \sqrt{E^2 - \left(1 - \frac{R_s}{r}\right)\frac{L^2}{r^2}}$$

$$\frac{dr}{dt} = \frac{dr/d\lambda}{d\lambda/dt}$$

$$\frac{dt}{d\lambda} = \frac{E}{1 - \frac{R_s}{r}}$$ from Killing equation

$$\frac{dr}{dt} = \left(1 - \frac{R_s}{r}\right)\sqrt{1 - \left(1 - \frac{R_s}{r}\right)\frac{L^2}{E^2 r^2}}$$

at r_0: $$\frac{dr}{dt} = 0 = \left(1 - \frac{R_s}{r_0}\right)\sqrt{1 - \left(1 - \frac{R_s}{r_0}\right)\frac{L^2}{E^2 r_0^2}}$$

$$\frac{L^2}{E^2} = \frac{r_0^2}{1 - \frac{R_s}{r_0}}$$

$$\frac{dr}{dt} = \left(1 - \frac{R_s}{r}\right)\sqrt{1 - \left(1 - \frac{R_s}{r}\right)\frac{r_0^2}{1 - \frac{R_s}{r_0}}}$$

$$\Delta t = \int_{r_0}^{d_E} \frac{dt}{dr}\, dr$$

doing some algebra, this integral is:

$$\Delta t = \int_{r_0}^{d_E} \frac{1 + \frac{R_s}{r}}{\sqrt{1 - \frac{r_0^2}{r^2}}}\left(1 - \frac{1}{2}\frac{R_s r_0}{r(r + r_0)}\right) dr$$

this integral evaluates to:

$$\boxed{\Delta t = \sqrt{d^2 - r_0^2} + R_s \ln\left[\frac{d + \sqrt{d^2 - r_0^2}}{r_0}\right] + \frac{1}{2} R_s \frac{\sqrt{d^2 - r_0^2}}{r_0 (d + r_0)}}$$

as stated earlier

Cool Things

what happens as we approach R_s?

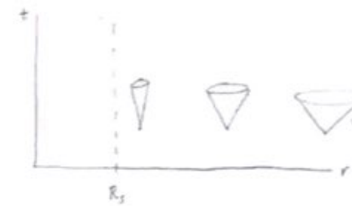
let's look at null trajectories in the $r-t$ plane:

$$ds^2 = 0$$

$$\left(1 - \frac{R_s}{r}\right) dt^2 = \frac{1}{1 - \frac{R_s}{r}}\, dr$$

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{R_s}{r}}$$

light cones



light cones encompass all points that are accessible to particles. Massive particles travel inside the cone. Massless particles travel on the surface of the cones.

let's change coordinates so that:

$$\frac{dt}{dr^*} = \pm 1$$

$$dr^* = \frac{dr}{1 - \frac{R_s}{r}}$$

$$r^* = r + R_s \ln\left[\frac{r}{R_s} - 1\right]$$

"tortoise coordinate"

$$r > R_s \Rightarrow -\infty < r^* < \infty$$

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right) dr^{*2}$$

note: $r = r(r^*)$

$$ds^2 = \left(1 - \frac{R_s}{r}\right)\left(-dt^2 + dr^{*2}\right)$$

thus, light cones in this coordinate system are the same everywhere now change coordinates again:

$$v = t + r^* \qquad dv = dt + dr^*$$
$$u = t - r^* \qquad du = dt - dr^*$$

"Eddington - Finkelstein coordinates"

$$du\,dv = dt^2 - dr^2$$

$$ds^2 = \left(1 - \frac{R_s}{r}\right)(-du\,dv) + r^2 d\Omega^2$$

now eliminate u:

$$du = dv - 2dr^*$$

$$du = dv - 2\frac{dr}{1 - \frac{R_s}{r}}$$

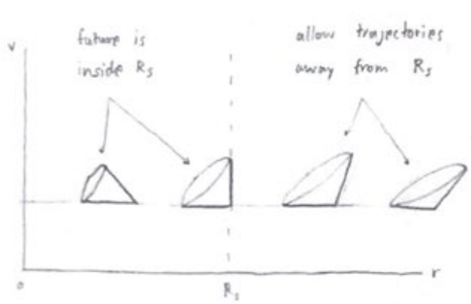
$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

our coordinates are: v, r, θ, φ

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{R_s}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

this is great, because now nothing goes wrong at $r = R_s$

light cones:



change coordinates yet again:

for $r > R_s$: "Kruskell coordinates"

$$r' = \sqrt{\frac{r}{R_s} - 1}\; e^{r/2R_s} \cosh\left(\frac{t}{2R_s}\right)$$

$$t' = \sqrt{\frac{r}{R_s} - 1}\; e^{r/2R_s} \sinh\left(\frac{t}{2R_s}\right)$$

for $r < R_s$:

$$r' = \sqrt{1 - \frac{r}{R_s}}\; e^{r/2R_s} \sinh\left(\frac{t}{2R_s}\right)$$

$$t' = \sqrt{1 - \frac{r}{R_s}}\; e^{r/2R_s} \cosh\left(\frac{t}{2R_s}\right)$$

for both regions:

$$r'^2 - t'^2 = \left(\frac{r}{R_s} - 1\right)e^{r/R_s}$$

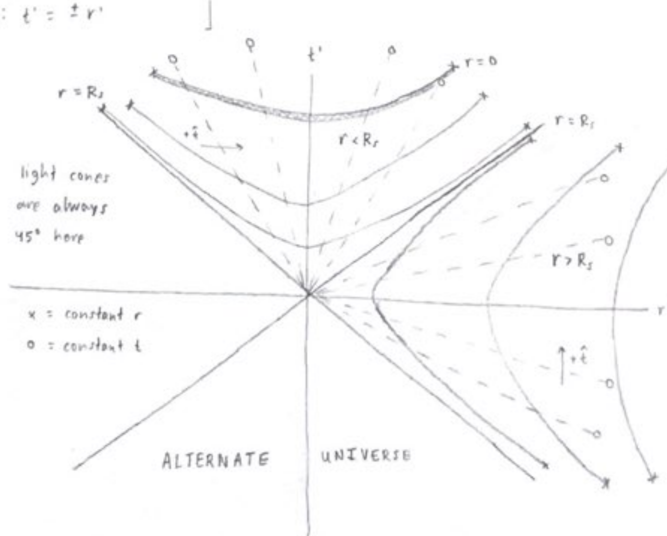
our metric is:

$$ds^2 = \frac{4R_s^3}{r} e^{-r/R_s}\left(-dt'^2 + dr'^2\right) + r^2 d\Omega^2$$

null lines $\frac{dt'}{dr'} = \pm 1$

$r > R_s$: $\frac{r'}{t'} = \coth\left(\frac{t}{2R_s}\right)$ lines of constant t
$r < R_s$: $\frac{r'}{t'} = \tanh\left(\frac{t}{2R_s}\right)$

$r > R_s$: $r'^2 - t'^2 = \text{(positive)}$ lines of constant r
$r < R_s$: $r'^2 - t'^2 = \text{(positive)}$
$r = R_s$: $t' = \pm r'$



Conformalism

$$\tilde{g}_{\mu\nu} = \omega^2(x)\, g_{\mu\nu}$$

conformally related if this transformation preserves angles

conformal transform of flat space:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

① switch coordinates to:

$$v = t + r \qquad dv = dt + dr$$
$$u = t - r \qquad du = dt - dr$$

$$ds^2 = -dv\,du + \frac{(v-u)^2}{4} d\Omega^2$$

② $v' = \tan^{-1}(v)$
$u' = \tan^{-1}(u)$

$$dv = \frac{dv'}{\cos^2 v'} \qquad du = \frac{du'}{\cos^2 u'}$$

$$(v-u)^2 = \left(\frac{\sin(v'-u')}{\cos v' \cos u'}\right)^2$$

$$ds^2 = \frac{1}{\cos^2 v' \cos^2 u'}\left(-du'dv' + \dots\right)$$

③ $t' = \frac{v'+u'}{2}$ $\quad dt'^2 = \frac{(dv'+du')^2}{4}$

$r' = \frac{v'-u'}{2}$ $\quad dr'^2 = \frac{(dv'-du')^2}{4}$

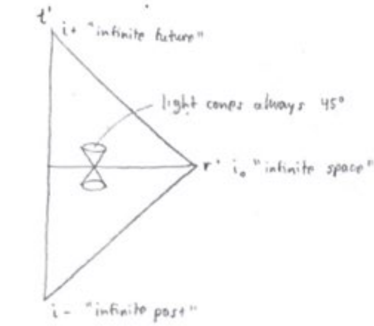
$$du'dv' = -dt'^2 + dr'^2$$

$$ds^2 = \omega^2(t',r')\left(-dt'^2 + dr'^2\right)$$

thus, this is a conformal transformation

note: $0 \le r' \le \pi$

$-\pi \le t' \le \pi$



Charged Black Hole

"reissner-nordstrom solution"

spherically-symmetric solution, $\vec{E} \sim \frac{E(r)}{r^2}\hat{e}_r$

the answer:

$$ds^2 = -\Delta dt^2 + \frac{1}{\Delta}dr^2 + r^2 d\Omega^2$$

$$\Delta = 1 - \frac{R_s}{r} + \frac{GQ^2}{r^2}$$

Q = charge of black hole

Cosmology

assume maximal symmetry in space but not in time

$$ds^2 = -dt^2 + R(t)^2 d\sigma^2$$

↑ scale factor, nothing to do with curvature

$d\sigma^2$ is maximally symmetric in 3-d space

now solve Einstein's equations under these assumptions

$$d\sigma^2 = e^{2B(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

note: in a space of maximal symmetry:

$$* \quad R_{\alpha\beta} = \frac{R}{d} g_{\alpha\beta}$$

additionally, we know:

$${}^{3D}R_{\bar{r}\bar{r}} = \frac{2}{\bar{r}}\partial_{\bar{r}} B$$

$${}^{3D}R_{\theta\theta} = e^{-2B}\left(\bar{r}\partial_{\bar{r}}B - 1\right) + 1$$

$${}^{3D}R_{\varphi\varphi} = \sin^2\theta \, R_{\theta\theta}$$

$d = 3$, say $R = 6K$ space of constant curvature

from *:

$$R_{\bar{r}\bar{r}} = 2K e^{2B} = \frac{2}{\bar{r}}\partial_{\bar{r}} B$$

$$e^{-2B}\partial_{\bar{r}} B = K\bar{r}$$

$$-\frac{1}{2}\partial_{\bar{r}}\left(e^{-2B}\right) = K\bar{r}$$

$$-\frac{1}{2}e^{-2B} = c + \frac{1}{2}K\bar{r}^2$$

$$B = -\frac{1}{2}\ln\left[c - K\bar{r}^2\right]$$

rescale coordinates so that $c \to 1$

$$e^{-2B} = 1 - K\bar{r}^2$$

so our metric for space is:

$$d\sigma^2 = \frac{1}{1-K\bar{r}^2}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

our full metric is:

$$ds^2 = -dt^2 + R^2(t)\left[\left(1-K\bar{r}^2\right)^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2\right]$$

change coordinates:

$$r = \sqrt{|K|}\,\bar{r}$$

$$ds^2 = -dt^2 + \frac{R(t)^2}{|K|}\left[\left(1-\xi r^2\right)^{-1}dr^2 + r^2 d\Omega^2\right]$$

$$\xi = \text{sign of } K : \begin{cases} 0 & \text{flat space} \\ 1 & \text{convex} \\ -1 & \text{concave} \end{cases}$$

call $a(t)^2 = \frac{R(t)^2}{|K|}$

the final form of our metric is:

$$\boxed{ds^2 = -dt^2 + a(t)^2\left[\left(1-\xi r^2\right)^{-1}dr^2 + r^2 d\Omega^2\right]}$$

this is the Robertson-Walker metric

More Cosmology

take a light ray from a distant star to us:

since it's a light ray:

$$ds^2 = 0$$

also, radial paths are geodesics, so we can say $d\Omega^2 = 0$

$$0 = -dt^2 + (1 - \varepsilon r^2)^{-1} dr^2 \, a(t)^2$$

calculate the time it takes:

$$\frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1 - \varepsilon r^2}}$$

$$\int_{t_{emit}}^{t_{observed}} \frac{1}{a(t)} \, dt = \int_0^{r_e} \frac{1}{\sqrt{1 - \varepsilon r^2}} \, dr$$

now imagine a ray sent slightly later:

$$t_{ob} \to t_{ob} + \delta t_{ob}$$

$$t_{em} \to t_{em} + \delta t_{em}$$

$$\int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{1}{a(t)} \, dt \to \int_{t_e}^{t_o} \frac{1}{a(t)} \, dt + \frac{\delta t_o}{a(t_o)} - \frac{\delta t_e}{a(t_e)}$$

we know, however, that:

$$\int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{1}{a(t)} \, dt = \int_{t_e}^{t_o} \frac{1}{a(t)} \, dt \qquad \text{so}$$

$$\frac{\delta t_o}{a(t_o)} - \frac{\delta t_e}{a(t_e)} = 0$$

$$\frac{\delta t_o}{\delta t_e} = \frac{a(t_o)}{a(t_e)}$$

$\frac{\delta t_o}{\delta t_e}$ can be interpreted as a doppler effect, which we can measure

it turns out that $\frac{\delta t_o}{\delta t_e} > 1$

meaning $\frac{a(t_o)}{a(t_e)} > 1$, so the universe is expanding! (+ all light is redshifted due to this expansion)

Massive Particles

how do massive particles propogate in the Robertson-Walker geometry?

generalize the killing equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad \Rightarrow \quad K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

suppose K is actually a tensor:

$$\nabla_\nu K_{\mu_1 \ldots \mu_n} + \nabla_{\mu_1} K_{\nu \ldots \mu_n} + \ldots + \nabla_{\mu_n} K_{\mu_1 \ldots \nu} = 0$$

in this case, our conserved quantity is:

$$K_{\mu_1 \ldots \mu_n} \frac{dx^{\mu_1}}{d\lambda} \cdots \frac{dx^{\mu_n}}{d\lambda} = \text{constant}$$

in the RW metric:

$$K_{\mu\nu} = a(t)^2 \left[g_{\mu\nu} + U_\mu U_\nu \right]$$

conserved quantity:

$$K_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = K^2$$

in the frame of the particle:

$$\frac{dx^\mu}{d\tau} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$K^2 = a(t)^2 \left[g_{00} + (V^0 U_0)^2 \right]$$

time component in the lab frame

$$K^2 = a(t)^2 \left[-1 + V^{0\,2} \right]$$

$$-V^{0\,2} + |\vec{v}|^2 = -1$$

3-velocity

$$K^2 = a(t)^2 |\vec{v}|^2$$

$$\boxed{\vec{v} = \frac{K}{a(t)}}$$

in our case (a is growing), all velocities decrease with time

Luminosity distance:

$$d_L^2 = \frac{L}{4\pi F}$$

luminosity

observed energy flux

change coords in RW:

$$d\chi = (1 - \varepsilon r^2)^{-1/2} dr$$

$$\chi = \begin{cases} \sin^{-1}(r) & \varepsilon = 1 \\ r & \varepsilon = 0 \\ \sinh^{-1}(r) & \varepsilon = -1 \end{cases}$$

RW metric becomes:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \chi_\xi^2 d\Omega^2\right]$$

$$\chi_\xi = \begin{cases} \sin\chi & \xi = 1 \\ \chi & \xi = 0 \\ \sinh\chi & \xi = -1 \end{cases}$$

$$\frac{F}{L} = \frac{1}{(1+z)^2 A}$$

$A = 4\pi \chi_\xi^2$ and $z+1 = \frac{1}{a(t)}$

z is the redshift of our light

One power of $z+1$ comes from the redshift, the second power comes from the counting rate of photons hitting our detector

suppose: $a(t) = 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 \ldots$

$H_0 = \dot{a}(t)$ "Hubble rate"

$$dt^2 = a^2(t)\, d\chi^2$$

$$\int_t^{t_0} \frac{1}{a(t)} dt = \int d\chi = \chi$$

$$z = \frac{1}{a} - 1$$

$$z = \left(1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2\right)^{-1} - 1$$

this approximates to:

$$z = H_0(t-t_0) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t-t_0)^2 \quad **$$

$$\chi = \int_t^{t_0} \frac{1}{1 + H_0(t-t_0)} dt'$$

$$\chi = (t_0 - t) + \frac{H_0}{2}(t_0 - t)^2 \quad *$$

we want:

$$\chi = \alpha z + B z^2 + \ldots$$

plug in * and **:

$$-(t-t_0) + \frac{H_0}{2}(t-t_0)^2 = \alpha H_0 (t-t_0)$$
$$+ \alpha\left(1 + \frac{q_0}{2}\right) H_0^2 (t-t_0)^2 + B H_0^2 (t-t_0)^2$$

keeping only second order terms:

examining this equation, we find:

$$\alpha H_0 = 1 \longrightarrow \alpha = \frac{1}{H_0}$$

$$\frac{H_0}{2} = \alpha\left(1 + \frac{q_0}{2}\right) H_0^2 + B H_0^2$$

$$\frac{1}{2} = \frac{1}{H_0}\left(1 + \frac{q_0}{2}\right) H_0 + B H_0$$

$$B = -\left(\frac{1 + q_0}{2 H_0}\right)$$

so we get:

$$\chi = \frac{1}{H_0} z - \left(\frac{1+q_0}{2H_0}\right) z^2 \ldots$$

plug this into luminosity distance:

$$d_L = \frac{z}{H_0}(1+z)\left(1 - \frac{1+q_0}{2} z\right)$$

$$\boxed{d_L = \frac{z}{H_0}\left(1 + \frac{1-q_0}{2} z\right)}$$

so q_0 is a measure of how much d_L differs from what we'd expect (which is that d_L scales linearly with z)

also: $\ddot{a} = -q_0 H_0^2 \quad \dot{a} = H_0$

$$q_0 = -\frac{\ddot{a}}{\dot{a}^2}$$

Finding $a(t)$

RW:

$$ds^2 = -dt^2 + a(t)^2\left[(1-\xi r^2)^{-1} dr^2 + r^2 d\Omega^2\right]$$

$$R_{tt} = -3\frac{\ddot{a}}{a}$$

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\xi}{1 - \xi r^2}$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\xi)$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$$

all off diagonal components are zero

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\xi}{a^2}\right)$$

assume $T_{\mu\nu}$ is that of a perfect fluid

$$T_{\mu\nu} = (p + \rho) U_\mu U_\nu + p g_{\mu\nu}$$

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p)$$

$$T = T^\mu{}_\mu = -\rho + 3p$$

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma$$

set $\nu = 0$ (energy conservation)

$$0 = \nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\sigma} T^\sigma{}_0 - \Gamma^\sigma_{\mu 0} T^\mu{}_\sigma$$

$$0 = -\partial_t \rho - 3\frac{\dot{a}}{a}(\rho + p)$$

rewrite this:

$$\left[\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)\right]$$

this tells us how the energy density changes depending on $a(t)$

it makes sense to assume:

$$p = w\rho$$

so the bracketed equation becomes:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + w\rho)$$

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1+w)$$

$$\Rightarrow \rho \sim a^{-3(1+w)}$$

for different types of matter:

	w	$-3(1+w)$
radiation	1/3	-4
matter	0	-3
cosmological constant	-1	0

plug into Einstein's eqs:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$

multiply by $g^{\mu\nu}$:

$$R - \frac{1}{2}R = 8\pi G\, T^{\mu}{}_{\mu}$$

$$R = -8\pi G T$$

rewrite Einstein:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

$$R_{tt} = 8\pi G\left(T_{tt} - \frac{1}{2}g_{tt}T\right)$$

$$\downarrow$$

$$-3\frac{\ddot{a}}{a} = 8\pi G\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)} \quad (1)$$

now do the rr component:

$$R_{rr} = 8\pi G\left[\frac{a^2}{1-kr^2}\left(p - \frac{1}{2}(-\rho + 3p)\right)\right]$$

$$\downarrow$$

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} = \frac{8\pi G}{1-kr^2}\left(\frac{1}{2}\right)a^2(\rho - p)$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

plug in $\frac{\ddot{a}}{a}$ from R_{tt}:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \quad (2)$$

(1) and (2) are called the Friedmann equations

Friedmann Eqs.

examine the second equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

recall that:

$$H_0 = \dot{a}\big|_{now}$$

we can normalize $a(now)$ to 1

$$H_0^2 = \frac{8\pi G}{3}\rho - k$$

call $\rho_{critical}$ s.t. $k = 0$:

$$\left[\rho_c = \frac{3H_0^2}{8\pi G}\right]$$

call Ω the density:

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3H_0^2}\rho$$

$$H_0^2 = \frac{8\pi G}{3}\rho - k$$

$$1 = \frac{8\pi G}{3H_0^2}\rho - \frac{k}{H_0^2}$$

$$1 = \Omega - \frac{k}{H_0^2}$$

$$\Omega = 1 + \frac{k}{H_0^2}$$

$$\left\{\begin{array}{l} \Omega = 1 \\ \Omega > 1 \\ \Omega < 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} k = 0 \\ k = 1 \\ k = -1 \end{array}\right\}$$

the density equation can be written for any time by rewriting H_0:

$$H_0 \rightarrow Ha$$

$$H = \frac{\dot{a}}{a} \Rightarrow H_0 \rightarrow \dot{a}$$

$$\Omega = 1 + \frac{k}{\dot{a}^2}$$

experimentally, we find:

$$H_0 = 69.3 \pm 0.8\ \frac{km/s}{mpc}$$

this implies that:

$$\rho_c = 9.2 \times 10^{-27}\ kg/m^3$$

furthermore, we've found $\rho = \rho_c \Rightarrow k = 0$

meaning our universe is flat

$H^2 = H_0^2 \sum \Omega_i - \frac{k}{a^2}$; i denotes matter, radiation, and Λ

$$H^2 = H_0^2 \left(\Omega_{m0} a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda - \Omega_k a^{-2} \right)$$

additionally:

$$-q_0 = \frac{\ddot{a}}{\dot{a}^2}\Big|_{now} = \frac{\ddot{a}}{H_0^2}$$

$$-q_0 = -\frac{4\pi}{3} \frac{G}{H_0^2} \sum \rho_i (1+3w_i)$$

recall: $w_m = 0$

$w_r = \frac{1}{3}$

$w_\Lambda = -1$

$\Omega_i = \frac{8\pi G}{3 H_0^2} \rho_i$ so

$$q_0 = \frac{1}{2} \sum_i \Omega_i (1+3w_i)$$

$$q_0 = \frac{1}{2}\left(\Omega_{m0} + 2\Omega_{r0} - 2\Omega_{\Lambda 0} \right)$$

in our universe, $\Omega_m = 0.3$

$\Omega_r = 0$

$\Omega_\Lambda = 0.7$

plugging this in, we get:

$\boxed{q_0 = 0.55}$ indicating that the expansion of the universe is accelerating

will the universe ever stop expanding?

expansion decelerates when $H = 0$

so is it the case that

$$H(t^*) = 0$$

for some t^*?

$$H^2 = H_0^2 \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_k a^{-2} \right)$$

ignoring Ω_r, we have:

$$\Omega_m a^{-3} + \Omega_\Lambda + \Omega_k a^{-2} = 0$$

$$\boxed{\Omega_m + \Omega_\Lambda a^3 + \Omega_k a = 0}$$

solutions to this equation lead to futures where the universe recollapses

Age of the universe?

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \qquad \dot{a} = Ha$$

$$\frac{da}{dt} = Ha$$

$$dt = \int_0^1 \frac{1}{H(a) \cdot a} \, da$$

0 = start of the universe

1 = now

we can plug in H from above and integrate to get:

$$t = \frac{1}{H_0} \int_0^1 \left[\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2 + \Omega_k \right]^{-1/2} da$$

$$t = \frac{2}{3} \frac{1}{H_0} \quad \text{(for matter domination)}$$

$$t = 0.96 \frac{1}{H_0} \quad (\text{for } \Omega_m = .3,\ \Omega_\Lambda = .7)$$

experimentally confirmed

$$\boxed{t = 13.4 \times 10^9 \text{ years}}$$

Inflationary Era

FRW in early stage (before Λ kicks in):

$$a(t) = t^q \qquad 0 < q < 1$$

$$q = \begin{cases} 2/3 & \text{matter domination} \\ 1/2 & \text{radiation domination} \end{cases}$$

$$ds^2 = -dt^2 + a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right]$$

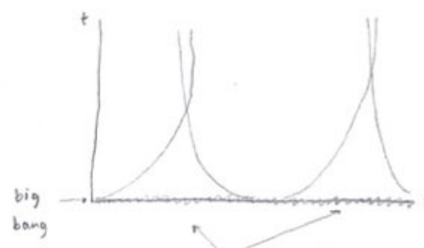
in our universe, we've observed $k = 0$

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2 d\Omega^2 \right]$$

for a photon: $ds^2 = 0$

$$0 = -dt^2 + a^2(t)\, dr \ldots$$

photon trajectories look like:



the two photons are causally disconnected

on our conformal diagram:

However, we've observed that the microwave background is in thermal equilibrium, even though its component photons come from causally disconnected places. This is the horizon problem.

solutions to the horizon problem?

$$a(t) = \left(\frac{t}{t_0}\right)^q$$

$$H_0 = \dot{a} = q\,\frac{t^{q-1}}{t_0^q}$$

$$\Delta r = \int_{t_1}^{t_2} \frac{1}{a(t)}\,dt$$

$$\Delta r = t_0^q\left(\frac{t_2^{1-q} - t_1^{1-q}}{1-q}\right)$$

$$r_{horizon} = \int_0^{t_H} \frac{1}{a(t)}\,dt$$

$$r_H = t_0^q\,\frac{t_H^{1-q}}{1-q}$$

$$r_H = \left(\frac{t_H}{t_0}\right)^{1-q}\frac{t_0}{1-q} = a(t_H)^{\frac{1-q}{q}}\left(\frac{q}{1-q}\right)\frac{1}{H_0}$$

for matter domination, $q = 2/3$:

$$r_H = \frac{1}{\sqrt{1200}}\,(2)\,\frac{1}{H_0}$$

$r_H = \frac{1}{17}\left(\frac{1}{H_0}\right)$ this says that pieces of the sky that are $\frac{1}{17}(2\pi)$ radians apart are causally disconnected... wrong

answer: inflation

$$r_H = \int_{t_0}^{t_i} \frac{1}{a(t)}\,dt$$

assume that for a short period of time:

$$a(t) = e^{\lambda t}$$

$$r_H = \frac{1}{\lambda}\left(e^{-\lambda t_0} - e^{-\lambda t_i}\right)$$

$d_{Horizon}$ is our physical scale

$$d_H = a(t_i)\,r_H$$

$$d_H = \frac{1}{\lambda}\left(e^{-\lambda t_0} - 1\right)$$

this says that all points of the observable cosmic background are causally connected

The Flatness Problem

recall:

$$\Omega_\varepsilon = 1 - \Omega_m - \Omega_r - \Omega_\Lambda$$

$$\Omega_\varepsilon = -\frac{\varepsilon}{H^2 a^2}$$

$$H^2 = H_0^2\left(\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\Lambda 0} + \Omega_{\varepsilon 0}a^{-2}\right)$$

$$\Omega_\varepsilon = \frac{\Omega_{\varepsilon 0}}{a^2\left(\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\Lambda 0} + \Omega_{\varepsilon 0}a^{-2}\right)}$$

$$\frac{1}{a} = 1 + z$$

$$\Omega_\varepsilon = \frac{\Omega_{\varepsilon 0}}{\Omega_{r0}(1+z)^2 + \Omega_{m0}(1+z) + \Omega_{\Lambda 0}(1+z)^{-2} + \Omega_{\varepsilon 0}}$$

Ω_r is negligible so $\Omega_\varepsilon \sim \frac{\Omega_{\varepsilon 0}}{z}$

a long time ago, $z = 1200$. Now, $z = 1$.

This means that even though the universe is flat now, it was even flatter at the time of the big bang.

This problem is also solved by inflation.

suppose Ω_Λ dominates during inflation:

$$\Omega_\varepsilon \sim \frac{\Omega_{\varepsilon 0}(1+z)^2}{\Omega_\Lambda}$$

$$\Omega_\varepsilon = \frac{\Omega_{\varepsilon 0}}{(1+z)^2\left(e^{\lambda t_0}\right)^2} \quad \text{end of inflation}$$

so basically:

$\Omega_\varepsilon = 0$ for all future times

Weak fields

assume $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

where $|h_{\mu\nu}| \ll 1$

$$\boxed{g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}}$$

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

proof:

$$g_{\mu\nu}\, g^{\nu\rho} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho})$$

$$= \delta_\mu^\rho + h_\mu{}^\rho - h_\mu{}^\rho + O(h^2)$$

$$= \delta_\mu^\rho \quad \checkmark$$

calculate the connections:

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}\left(g_{\nu\rho,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu}\right)$$

to $O(h)$:

$$\boxed{\Gamma^\mu_{\rho\sigma} = \frac{1}{2}\eta^{\mu\nu}\left(h_{\nu\rho,\sigma} + h_{\sigma\nu,\rho} - h_{\rho\sigma,\nu}\right)}$$

calculate the curvature tensor:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + O(\Gamma^2)$$

plug in the connection:

$$R^\mu{}_{\nu\rho\sigma} = \frac{1}{2}\eta^{\mu\lambda}\left(h_{\lambda\nu,\sigma\rho} + h_{\lambda\sigma,\nu\rho} - h_{\nu\sigma,\lambda\rho} - h_{\lambda\nu,\sigma\rho} + h_{\lambda\rho,\nu\sigma} + h_{\nu\rho,\lambda\sigma}\right)$$

conveniently, terms cancel:

$$\boxed{R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right)}$$

the ricci tensor is, therefore:

$$\boxed{R_{\nu\sigma} = \frac{1}{2}\left(h^\mu{}_{\sigma,\mu\nu} + h^\mu{}_{\nu,\mu\sigma} - \Box h_{\nu\sigma} - h^\mu{}_{\mu,\nu\sigma}\right)}$$

where $\Box$ is the d'Alenbertian

$$\Box = \partial_\mu \partial^\mu$$

finally, the ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu}$$

in our case $g^{\mu\nu} \sim \eta^{\mu\nu}$

$$R = \eta^{\mu\nu} R_{\mu\nu} \qquad h = h^\mu{}_\mu$$

$$\boxed{R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h}$$

which implies:

$$\boxed{R_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma} R = \frac{1}{2}\left(h^\mu{}_{\sigma,\mu\nu} + h^\mu{}_{\nu,\sigma\mu} - \Box h_{\sigma\nu} - h_{,\sigma\nu} + \eta_{\nu\sigma} h^{\mu\lambda}{}_{,\mu\lambda} + \eta_{\nu\sigma}\Box h\right)}$$

what if we change coordinates?

$$x'^\mu = x^\mu + \Lambda(x)^\mu$$

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} g^{\rho\sigma}$$

$$\eta^{\mu\nu} - h'^{\mu\nu} = \left(\delta^\mu_\rho + \frac{\partial \Lambda^\mu}{\partial x^\rho}\right)\left(\delta^\nu_\sigma + \frac{\partial \Lambda^\nu}{\partial x^\sigma}\right) g^{\rho\sigma}$$

$$g^{\rho\sigma} = \eta^{\rho\sigma} - h^{\rho\sigma}$$

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \Lambda_\mu}{\partial x^\nu} - \frac{\partial \Lambda_\nu}{\partial x^\mu}$$

assume $\left|\frac{\partial \Lambda^\mu}{\partial x^\nu}\right| \ll 1$, then

$$R_{\mu\nu\rho\sigma}(h') = R_{\mu\nu\rho\sigma}(h)$$

this is a guage transformation to which the curvature is invarient

More on $h_{\mu\nu}$

since $h_{\mu\nu}$ is symmetric, it has 10 DOFs

$$h_{\mu\nu} = \begin{array}{c|c} h_{00} & h_{0i} \\ \hline h_{i0} & h_{ij} \end{array}$$

under rotation, h_{00} transforms like a scalar, h_{0i} like a vector, and h_{ij} like a we can decompose $h_{\mu\nu}$ into two scalar fields, a vector field, and a tensor field

$$h_{00} = -2\phi \quad \text{(scalar)}$$

$$h_{0i} = h_{i0} = w_i \quad \text{(vector)}$$

$$h_{ij} = 2(s_{ij} - \delta_{ij}\underbrace{\Psi}_{\text{(scalar)}}) \quad \text{(tensor)}$$

similarly, we can write the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$G_{00} = 2\nabla^2 \Psi + \partial_i \partial_j s^{ij}$$

we can similarly represent G_{0i} and G_{ij}

assume static sources of energy:

$$T_{\mu\nu} = U_\mu U_\nu \rho \qquad U_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$T_{00} = \rho$ all other $T_{\mu\nu} = 0$

Einstein's equations say:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$2\nabla^2\phi + \partial_i\partial_j s^{ij} = 8\pi G\rho$$

use gauge transforms to eliminate $\partial_i \partial_j s^{ij}$:

$$2\nabla^2\phi = 8\pi G\rho$$

in the limit: $\phi = \Psi$, so:

$$\boxed{g_{\mu\nu} = -(1+2\phi)\,dt^2 + (1-2\phi)\,d\vec{x}^{\,2}}$$

Waves

warm up with EM waves in a vacuum:

$$\partial^\mu F_{\mu\nu} = 0$$

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0$$

pick the lorentz gauge: $\partial^\mu A_\mu = 0$

$$\Box A_\nu = 0$$

assume:

$$A_\nu = \varepsilon_\nu e^{ikx} + \varepsilon_\nu^* e^{-ikx}$$

$$\Box A_\nu = -k^2(\varepsilon_\nu e^{ikx} + \varepsilon_\nu^* e^{ikx})$$

$$\Rightarrow k^2 = 0 \Rightarrow E^2 - \vec{p}^{\,2} = 0$$

(since k is a four vector) $\Rightarrow$ our waves are massless and move at the speed of light

note: ε_ν is the polarization vector

gravitational waves:

"harmonic gauge": $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$

$$\Rightarrow h^\mu{}_{\lambda,\mu} = \frac{1}{2} h_{,\lambda}$$

wave equation:

$$R_{\sigma\nu} = 0$$

$$\frac{1}{2}(h^\mu{}_{\nu,\sigma\mu} + h^\mu{}_{\sigma,\nu\mu} - \Box h_{\sigma\nu} - h_{,\sigma\nu}) = 0$$

using the lorentz gauge, this reduces to:

$$\Box h_{\mu\nu} = 0$$

assume plane wave solution:

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ikx} + \varepsilon_{\mu\nu}^* e^{-ikx}$$

$$\Box h_{\mu\nu} \Rightarrow k^2 = 0$$

this means that gravitational waves move at the speed of light

now impose the harmonic gauge:

$$h^\mu{}_{\nu,\mu} = \frac{1}{2} h_{,\nu}$$

$$k_\mu \varepsilon^\mu{}_\nu = \frac{1}{2} k_\nu \varepsilon^\mu{}_\mu$$

our gauge transformation is:

$$h_{\mu\nu}' = h_{\mu\nu} - \partial_\mu \chi_\nu - \partial_\nu \chi_\mu$$

$$\chi_\mu = i q_\mu (e^{ikx} - e^{-ikx})$$

$$\Rightarrow \varepsilon_{\mu\nu}' = \varepsilon_{\mu\nu} + k_\mu q_\nu + k_\nu q_\mu$$

check that this satisfies the harmonic gauge:

$$k_\mu \varepsilon^\mu{}_\nu{}' = k_\mu \varepsilon^\mu{}_\nu + \cancel{k_\mu k^\mu} q_\nu + k_\nu k^\mu q_\mu$$

$= \frac{1}{2} k_\nu \varepsilon^\mu{}_\mu$? Yep, apparently

say $k_\mu = E(1,0,0,1)$

$$k_\mu \varepsilon^\mu{}_\nu = \frac{1}{2} k_\nu \varepsilon^\mu{}_\mu$$

for $\nu = 0$:

$$-\varepsilon_{00} + \varepsilon_{03} = \frac{1}{2}(-\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

$\nu = 3$: $-\varepsilon_{03} + \varepsilon_{33} = \frac{1}{2}(-\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

$\nu = 2$: $-\varepsilon_{01} + \varepsilon_{31} = 0$

$\nu = 1$: $-\varepsilon_{02} + \varepsilon_{32} = 0$

So we know that:

$$\varepsilon_{01} = \varepsilon_{31}$$

$$\varepsilon_{02} = \varepsilon_{32}$$

by other linear manipulations, we get:

$$\varepsilon_{03} = \frac{1}{2}\left(\varepsilon_{00} + \varepsilon_{33}\right)$$

$$\varepsilon_{22} = -\varepsilon_{11}$$

choose q_0 and q_3 to set:

$$\varepsilon_{00} = \varepsilon_{33} = 0$$

choose q_1 and q_2 to set:

$$\varepsilon_{01} = \varepsilon_{02} = 0$$

$$\Rightarrow \varepsilon_{31} = \varepsilon_{32} = 0$$

so we've reduced $\varepsilon_{\mu\nu}$ to:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so we have two propagating degrees of freedom

for convenience: call $\varepsilon_{11} = h_+$

$\varepsilon_{12} = h_\times$

so our propagation matrix is:

$$\begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}$$

$T_{\mu\nu}$ for Waves

to first order in $h_{\mu\nu}$:

$$R^{(1)}_{\alpha\beta} = \frac{1}{2}\left(h^{\mu}{}_{\alpha,\beta\mu} + h^{\mu}{}_{\beta,\alpha\mu} - \Box h_{\alpha\beta} - h^{\mu}{}_{\mu,\alpha\beta}\right)$$

we're solving Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

add $R^{(1)}$ to both sides and rearrange:

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = 8\pi G \Big[T_{\mu\nu} + \underbrace{\frac{1}{8\pi G}\left(R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} - R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R\right)}_{t_{\mu\nu}\text{, the gravitational energy-momentum tensor}} \Big]$$

we have:

$$\frac{\partial}{\partial x^\alpha}\left(R^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} R\right) = 0$$ (all terms in the derivative cancel)

so:

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = 8\pi G\left(T_{\mu\nu} + t_{\mu\nu}\right)$$

$$\Rightarrow \partial_\mu\left(T_{\mu\nu} + t_{\mu\nu}\right) = 0$$

$h^2 + h^3 + h^4 \ldots$

now let's calculate $t_{\mu\nu}$:

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(-R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)}\right)$$

$R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + R^{(3)}_{\mu\nu} \ldots$

$$t_{\mu\nu} = \frac{1}{8\pi G}\Big(-R^{(2)}_{\mu\nu} - R^{(3)}_{\mu\nu} - R^{(4)}_{\mu\nu} \ldots + \frac{1}{2}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\left(R^{(1)} + R^{(2)} \ldots\right) + R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)}\Big)$$

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(-R^{(2)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} R^{(2)} + \frac{1}{2} h_{\mu\nu} R^{(1)}\right)$$

to quadratic order

but we solved $R^{(1)}_{\mu\nu} = 0 \Rightarrow R^{(1)} = 0$, so:

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(-R^{(2)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} R^{(2)}\right)$$

from bracketed eqs. in "weak fields":

$$R^{(2)}_{\beta\delta} = -\frac{1}{2} h^{\alpha\sigma} R^{(1)}_{\alpha\sigma\beta\delta} + \eta_{\alpha\sigma}\left(\Gamma^{\sigma}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\beta} - \Gamma^{\sigma}_{\delta\lambda}\Gamma^{\lambda}_{\alpha\beta}\right)$$

expanded, this has a lot of $h_{\mu\nu}$ terms. Next, plug in our wave solution:

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik_\alpha x^\alpha} + \varepsilon^*_{\mu\nu} e^{-ik_\alpha x^\alpha}$$

$$\langle R^{(2)}_{\beta\delta} \rangle = \frac{1}{2} k_\beta k_\delta \left(\varepsilon^{*\mu\nu}\varepsilon_{\mu\nu} - \frac{1}{2}\varepsilon^{*\mu}{}_{\mu}\varepsilon^{\nu}{}_{\nu}\right)$$

$$\langle R^{(2)} \rangle = 0$$

this implies that:

$$\boxed{t_{\mu\nu} = \frac{-k_\mu k_\nu}{16\pi G}\left(\varepsilon^{*\mu\nu}\varepsilon_{\mu\nu} - \frac{1}{2}\varepsilon^{*\mu}{}_{\mu}\varepsilon^{\nu}{}_{\nu}\right)}$$

$$t_{\mu\nu} = \frac{-k_\mu k_\nu}{16\pi G}\left(h_+^2 + h_\times^2\right)$$

$$k^\mu = \omega\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$t_{03} = \frac{\omega^2}{8\pi G}\left(h_+^2 + h_\times^2\right)$$

Waves with Source

E-M: $\partial_\mu F^{\mu\nu} = J^\nu$

J^ν is the charge and current source

in the lorentz guage: $\partial_\mu A^\mu = 0$:

$$\Box A^\nu = J^\nu$$

Gravitation:

in the harmonic guage:

$$-\frac{1}{2}\Box h_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu} T\right)$$

we want to solve:

$$\Box \Psi(\vec{x},t) = 4\pi f(x,t)$$

it suffices to find the Green's function:

$$\Box G(x,t,x',t') = -4\pi \delta^3(x-x')\delta(t-t')$$

do fourier transform:

$$\Psi(x,\omega) = \int \Psi(x,t)e^{i\omega t}\,dt$$

$$\Psi(x,t) = \frac{1}{2\pi}\int \Psi(x,\omega)e^{-i\omega t}\,d\omega$$

$$(-\partial_t^2 + \nabla^2)\int G(x,\omega,x',\omega')e^{-i\omega t}\,d\omega$$

$$= -4\pi\,\delta^3(x-x')\frac{1}{2\pi}\int e^{i\omega(t'-t)}\,d\omega$$

this is equivalent to the equation:

$$(\nabla^2 + \omega^2)G_\omega(x-x') = -4\pi\,\delta^3(x-x')$$

switch to spherical:

$$x - x' = R$$

we get:

$$\frac{1}{R}\frac{d^2}{dR^2}\left(R\,G_\omega(R)\right) + \omega^2 G_\omega(R) = 4\pi\,\delta(R)e^{i\omega t}$$

when $R \neq 0$:

$$\frac{d^2}{dR^2}\left(R\,G_\omega(R)\right) + \omega^2 R\,G_\omega(R) = 0$$

this is just the harmonic oscillator, which agrees with the wave solution we found in a vacuum

$$R\,G_\omega(R) = e^{\pm i\omega R}$$

$$G_\omega(R) = \frac{1}{R}e^{\pm i\omega R}$$

this is the equation for plane waves in spherical coordinates

when $R = 0$:

$$G^{\pm}(x-x',t-t') = \frac{1}{2\pi}\int \frac{1}{|x-x'|}e^{\pm i\omega|x-x'|}e^{i\omega(t-t')}\,d\omega$$

$$= \frac{1}{2\pi}\frac{1}{|x-x'|}\int e^{i\omega(t-t'\pm|x-x'|)}\,d\omega$$

$$G^{\pm} = \frac{\delta(t'-t\pm|x-x'|)}{|x-x'|}$$

we're trying to find the gravitational wave at (x,t) emmitted by a source at (x',t')

$$\Psi(x,t) = \int G^+(x,t,x',t')f(x',t')\,d^3x'\,dt'$$

in our case:

$$h_{\mu\nu} = -4\pi G\int G^+(x,t,x',t')S_{\mu\nu}(x',t')\,d^3x'\,dt'$$

where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$

$$h_{\mu\nu} = -4\pi G\int \frac{\delta(t'-t+|x-x'|)}{|x-x'|}S_{\mu\nu}(x',t')\,d^3x'\,dt'$$

$$\boxed{h_{\mu\nu} = -4\pi G\int \frac{S_{\mu\nu}(x',t-|x-x'|)}{|x-x'|}\,d^3x'}$$

this gives us $h_{\mu\nu}$ as a function of $S_{\mu\nu}$.

Usually this integral is hard to solve.

we want to know how much energy is emmitted per solid angle

assume: $S_{\mu\nu} = S_{\mu\nu}(x,\omega)e^{-i\omega t}$ + complex conjugate (cc)

$$h_{\mu\nu} = -4\pi G\int \frac{S_{\mu\nu}(x,\omega)e^{-i\omega(t-|x-x'|)}}{|x-x'|}\,d^3x' + cc$$

in the radiation zone: $|x| \gg |x'|$

$|x|$ = distance from source

$|x'|$ = size of source

$|x'| \ll \frac{1}{\omega}$ (the wavelength)

$$|x-x'| = \sqrt{\underbrace{\vec{x}^2}_{r^2} + \underbrace{\vec{x}'^2}_{\text{small}^2} - 2\vec{x}\cdot\vec{x}'}$$

$$= r\sqrt{1 - 2\frac{x\cdot x'}{r^2}} = r - \hat{x}\cdot x'$$

$$h_{\mu\nu} = -4\pi G\frac{e^{ik_\mu x^\mu}}{r}\int S_{\mu\nu}(x',\omega)e^{-i\vec{k}\cdot\vec{x}'}\,d^3x'$$

$$h_{\mu\nu} = -4\pi G\frac{e^{ikx}}{r}S_{\mu\nu}(\vec{k},\omega)$$

Power

$$\frac{dP}{d\Omega} = r^2 \hat{x}_i \cdot T^{0i}$$

recall that:

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{16\pi G}\left[\epsilon^{*\mu\nu}\epsilon_{\mu\nu} - \frac{1}{2}|\epsilon|^2\right]$$

$$\frac{dP}{d\Omega} = \frac{r^2 \hat{x}\cdot\vec{k}\omega}{16\pi G}\left[\epsilon^{*\mu\nu}\epsilon_{\mu\nu} - \frac{1}{2}|\epsilon|^2\right]$$

$$\vec{k} = \omega\begin{pmatrix}1\\ \hat{x}\end{pmatrix} \quad \text{so} \quad \omega = k^0$$

for some reason: (1)

$$\epsilon^{*\mu\nu}\epsilon_{\mu\nu} - \frac{1}{2}|\epsilon|^2 = \left(\frac{4G}{r}\right)^2\left[S^{*\mu\nu}S_{\mu\nu} - \frac{1}{2}|S|^2\right]$$

so we get:

$$\frac{dP}{d\Omega} = \frac{\omega^2 G}{\pi}\left[S^{*\mu\nu}S_{\mu\nu} - \frac{1}{2}|S|^2\right]$$

(1) explained because we defined

$$h_{\mu\nu} = \epsilon^*_{\mu\nu}e^{ikx} + \epsilon_{\mu\nu}e^{-ikx}$$

comparing with:

$$h_{\mu\nu} = -\frac{4G}{r}S_{\mu\nu}e^{ikx} + cc$$

we know:

$$\epsilon_{\mu\nu} = -\frac{4G}{r}S_{\mu\nu}$$

in terms of $T_{\mu\nu}$:

$$\boxed{\frac{dP}{d\Omega} = \frac{\omega^2 G}{\pi}\left[T^{*\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right]}$$

we can write this in terms of spatial components using conservation laws:

$$\partial_\mu T^{\mu\nu}(k,\omega) = 0$$

$$T^{\mu\nu}(\vec{x},t) = \int T(k,\omega)e^{ikx}\frac{d^4k}{(2\pi)^4}$$

$$\partial_\mu T^{\mu\nu}(x,t) = \int k_\mu T^{\mu\nu}(k,\omega)\frac{d^4k}{(2\pi)^4}$$

$$k_\mu T^{\mu\nu} = 0 \Rightarrow \left.\begin{aligned} T^{00} &= \hat{k}_i\hat{k}_j T^{ij} \\ T^{0j} &= \hat{k}_i T^{ij}\end{aligned}\right]*$$

$$\text{where } \hat{k}_i = \frac{k_i}{\omega}$$

so the bracketed term becomes:

$$[\ ] = T^{*00}T^{00} - 2T^{*0i}T^{0m}\delta_{im} + T^{ij}T^{mn}\delta_{in}\delta_{jn} - \frac{1}{2}|-T^{00} + T^{ij}\delta_{ij}|^2$$

replace any instance of T^{00} or T^{0j} with the starred relations above

$$[\ ] = \left(\frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n - 2\hat{k}_j\hat{k}_n\delta_{im} + \delta_{im}\delta_{jn} + \frac{1}{2}\hat{k}_i\hat{k}_j\delta_{mn} + \frac{1}{2}\delta_{ij}\hat{k}_m\hat{k}_n\right)T^{ij}T^{mn}$$

we've eliminated all time (zero) components!

call the parenthesis term: $\Lambda_{ij,lmn}$

$$\frac{dP}{d\Omega} = \frac{\omega^2 G}{\pi}\Lambda_{ij,lmn}T^{ij}T^{mn}$$

Quadrupole Approx.

make the non-relativistic approximation:

$$d\omega \ll 1$$

$$T^{\mu\nu}(k,\omega) = \int T^{\mu\nu}(\vec{x},\omega)e^{i\vec{k}\cdot\vec{x}}\,d^3x$$

dipole approx. drop this

$$\rightarrow \int T^{ij}(x,\omega)\,d^3x$$

i.e. all variation comes from the time domain

now express everything in terms of T^{00}

for a single frequency:

$$T^{\mu\nu}(x,t) = T^{\mu\nu}(x,\omega)e^{-i\omega t} + cc$$

$$\partial_\mu T^{\mu\nu} = \partial_i T^{i\nu} + \partial_0 T^{0\nu}$$

$$= \partial_i T^{i\nu} - i\omega T^{0\nu} = 0$$

$$\text{so } \partial_i T^{i0} = i\omega T^{00}$$

$$\partial_i T^{ij} = i\omega T^{0j}$$

$$\partial_i\partial_j T^{ij} = i\omega\,\partial_j T^{0j} = -\omega^2 T^{00}$$

$$\underbrace{\int x^m x^n \partial_i\partial_j T^{ij}\,d^3x}_{\text{integrate by parts}} = -\omega^2\int x^m x^n T^{00}\,d^3x$$

$$\int(\delta_i^m\delta_j^n + \delta_i^n\delta_j^m)T^{ij}\,d^3x = -\omega^2\int x^m x^n T^{00}\,d^3x$$

$$\int T^{mn}\,d^3x = -\omega^2\int x^m x^n T^{00}\,d^3x$$

sub this into: $T^{\mu\nu}(k,\omega) = \int T^{ij}(x,\omega)\,d^3x$

$$T^{\mu\nu}(k,\omega) = -\omega^2\int x^m x^n T^{00}\,d^3x$$

$$T^{\mu\nu}(k,\omega) = -\omega^2 \underbrace{\int x^m x^n T^{00} d^3x}_{Q_{mn}}$$

$$\left[\frac{dP}{d\Omega} = \frac{\omega^2 G}{\pi} \Lambda_{ij|mn} Q^{*ij}(\omega) Q^{mn}(\omega) \left(\frac{\omega^2}{2}\right)^2 \right]$$

integrate over Ω to get total power radiated:

first: $\int d\Omega = 4\pi$

$$\int \hat{k}_i \hat{k}_j \, d\Omega = \frac{4\pi}{3} \delta_{ij}$$

cool trick to find $\frac{4\pi}{3}$ term:

assume $\int \hat{k}_i \hat{k}_j \, d\Omega = c\, \delta_{ij}$

take trace of both sides:

$$\int \hat{k}^2 \, d\Omega = 3c$$

$\hat{k}^2 = 1$ be unit vector

so $c = \frac{4\pi}{3}$

$$\int \hat{k}_i \hat{k}_j \hat{k}_m \hat{k}_n \, d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right)$$

$$P = \frac{\omega^6 G}{4\pi} Q^{*ij} Q^{mn} \int \Lambda_{ijmn} \, d\Omega$$

using our nice new integrals, this evaluates to:

$$\boxed{P = \frac{2}{5} G\omega^6 \left[Q^{*ij} Q_{ij} - \frac{1}{3} Q^2 \right] \qquad Q^{ij}(\omega) = \int x^i x^j \, T^{00}(x,\omega)\, d^3x}$$

this is the quadrupole approximation

Applications

Rigid-body rotation:

$$J_{ij} = \int x_i' x_j' \, \rho(x') \, d^3x'$$

$\uparrow$ mass density

this is the moment of inertia tensor

suppose: $x(t) = x'\cos(\Omega t) - y'\sin(\Omega t)$

$y(t) = y'\cos(\Omega t) + x'\sin(\Omega t)$

$z(t) = z'$

$$Q_{xx} = \int d^3x \; x(t)^2 \, \overset{T^{00}(x,\omega)}{\rho(x)}$$

switch to primed coordinates:

$$Q_{xx} = \int d^3x' \left(x'\cos(\Omega t) - y'\sin(\Omega t)\right)^2 \rho(x')$$

$$= J_{xx}\cos^2(\Omega t) + J_{yy}\sin^2(\Omega t)$$

$$- 2 J_{xy} \sin(\Omega t)\cos(\Omega t)$$

pick our rotation axis s.t. J is diagonal

$\rightarrow J_{xy} = 0$ so:

$$Q_{xx} = J_{xx}\cos^2(\Omega t) + J_{yy}\sin^2(\Omega t)$$

$$= J_{yy} + (J_{xx} - J_{yy})\cos^2(\Omega t)$$

$$= \underbrace{\tfrac{1}{2}(J_{xx} + J_{yy})}_{Q_{xx}(0),\ \text{set to zero}} + \tfrac{1}{2}(J_{xx} - J_{yy})\cos(2\Omega t)$$

so we get:

$$Q_{xx}(\overset{\omega = 2\Omega}{2\Omega}) = \frac{1}{4}(J_{xx} - J_{yy})$$

similarly

$$Q_{yy}(2\Omega) = \frac{1}{4}(J_{yy} - J_{xx})$$

$$Q_{xy}(2\Omega) = \frac{1}{4}(J_{xx} - J_{yy})$$

calculate the power emmitted:

$$P = \frac{2}{5} G (2\Omega)^6 \left[|Q_{xx}|^2 + |Q_{yy}|^2 + 2|Q_{xy}|^2 - \frac{1}{3}|Q_{xx} + Q_{yy}|^2 \right]$$

$$\boxed{P = \frac{32}{5} G\Omega^6 (J_{xx} - J_{yy})^2}$$

this is the power emmitted by a rigid-body rotating at angular velocity Ω

Binary system with distance R and reduced mass m (circular orbits):